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On oscillations in turbulent accretion disks: I. A general approach

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ABSTRACT
In this note, a general theory of a wave propagation in a turbulent media is reviewed. The main attention is paid to the case of the rotating flows with a weak magneto-hydrodynamic turbulence and vanishing mean magnetic flux. We derive the inhomogeneous wave equation and identify the excitation and damping terms. As a consequence of a weak turbulence, these terms can be treated as perturbations and the oscillations of the turbulent media can be decomposed into the normal modes of the laminar mean flow. Using the perturbation techniques we estimate the amplitudes of excited oscillations and show that their frequencies are shifted with respect to the case of the laminar flow.

Keywords: black hole physics – accretion disks – oscillations – turbulence

1 INTRODUCTION

According to a widely accepted scenario, the transport of angular momentum in accretion disks is accomplished by the magneto-hydrodynamic (MHD) turbulence driven by the magneto-rotational instability (MRI). On the other hand, in many studies of the accretion disk oscillations, the effects of turbulence are being neglected. It is not clear whether such idealized hydrodynamic perturbation of laminar background flow would survive also in highly turbulent magnetized media typical for real accretion disks.

Several attempts to resolve this issue using numerical simulations have lead to different conclusions. Brandenburg (2005) explored this problem in local shearing sheet simulations with negative result – his power spectra of time variability did not show any discrete frequencies. On the other hand, Arras et al. (2006) performed similar simulations and found an evidence for excitation of the radial epicyclic mode and various acoustic p-mode oscillations, but they did not find any evidence for inertial waves. Likely, differences between these results can be attributed to different strength of MHD turbulence – the zero net-magnetic-flux simulations by Arras et al. (2006) give rise to the weakest MRI-driven turbulence possible. There are also several global simulations partially devoted to this problem. For example, Kato (2004) studied radial and vertical oscillations in a thin MHD accretion flow. In his work, two pairs of oscillations are present in the region between 3.8 and 6.3 Schwarzschild radii. One of the oscillations could be identified as the radial $m = 1$ epicyclic mode excited in a resonance with the orbital motion. Contrary to this result, there is no evidence for such resonance in another global simulation of Reynolds and Miller (2009), but they agree with Arras et al. (2006) on the absence of inertial modes.

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Another way how to tackle this problem is to use the semi-analytical methods originally developed for solving similar problems in stars and Sun. The original idea by Lighthill (1952) was first used by Goldreich and Keeley (1977) in calculations of excitation/damping of solar p-modes by turbulent convection and it was later significantly developed by Goldreich and Kumar (1990); Samadi and Goupil (2001) and others (see Houdek, 2006, for a review). In this approach, the key role is plaid by so-called inhomogeneous wave equation (IWE) derived from the set of hydrodynamic equations by keeping also terms that are nonlinear in perturbations. While the homogeneous part that is linear in perturbations describes the oscillations of the laminar mean flow, the effects of the turbulence are included in the extra source terms on the right-hand side of the equation. From the mathematical point of view, IWE is a stochastic partial differential equation that can be translated to a set of stochastic ordinary differential equations if the oscillations can be decomposed into the normal modes.

In this paper we adopt this approach to the case of the differentially rotating unmagnetized flow with weak MHD turbulence. This setup is perhaps suitable for the accretion disks. In Section 2 we obtain IWE from the set of the nonlinear MHD equations. A decomposition into the normal modes is done in Section 3. We also derive the ordinary differential equations describing the instantaneous amplitudes of individual modes. Approximate solutions of these equations is found in Section 4. Finally, Section 5 is devoted to a discussion and conclusions.

2 INHOMOGENEOUS WAVE EQUATION

We generalize the approach of Goldreich and Keeley (1977) by considering MHD turbulence on a stationary (i.e. non-static) unmagnetized background flow. The evolution of the system is described by the continuity equation, induction equation and Euler equation,

$$\frac{\partial \rho}{\partial t} + \nabla_k\left(\rho v^k\right) = 0\,, \tag{1}$$

$$\frac{\partial B^i}{\partial t} + \nabla_k\left(B^i v^k - B^k v^i\right) = 0\,, \tag{2}$$

$$\frac{\partial}{\partial t}\left(\rho v^i\right) + \nabla_k\left(\rho v^i v^k\right) + \rho\nabla^i\Phi + \nabla^i\left(p + \frac{B^2}{8\pi}\right) - \frac{1}{4\pi}\nabla_k\left(B^k B^i\right) = 0\,, \tag{3}$$

together with the barotropic equation of state, $p = p(\rho)$, and the solenoidal condition $\nabla_k B^k = 0$. We assume that there exist a stationary and axisymmetric configuration with vanishing magnetic field, obtained from the above equations by setting $B^i = \partial_t = \partial_\phi = 0$, with a smooth velocity field describing a pure rotation $v^i = \Omega(r, z)\delta^i_\phi$ expressed in the cylindrical coordinates $\{r, \phi, z\}$. The *exact* equations describing perturbations of this equilibria are

$$\frac{\partial \delta\rho}{\partial t} + v^k\nabla_k\delta\rho + \nabla_k\left(\rho\delta v^k\right) = \mathcal{N}_\rho\,, \tag{4}$$

$$\frac{\partial \delta B^i}{\partial t} + v^k \nabla_k \delta B^i - \left(\nabla_k v^i\right) \delta B^k = \mathcal{N}_B^i \,, \tag{5}$$

$$\rho \left[\frac{\partial \delta v^i}{\partial t} + v^k \nabla_k \delta v^i + \left(\nabla_k v^i\right) \delta v^k + \nabla^i \left(c_{\mathrm{s}}^2 \frac{\delta \rho}{\rho} \right) \right] = \mathcal{N}_v^i \tag{6}$$

in which the nonlinear terms in perturbations on the right-hand sides are

$$\mathcal{N}_\rho = -\nabla_k \left(\delta\rho \delta v^k\right) , \tag{7}$$

$$\mathcal{N}_B^i = -\nabla_k \left(\delta v^k \delta B^i - \delta v^i \delta B^k\right) , \tag{8}$$

$$\begin{aligned} \mathcal{N}_v^i = &- \frac{\partial}{\partial t}\left(\delta\rho\delta v^i\right) - v^k \nabla_k \left(\delta\rho\delta v^i\right) - \left(\nabla_k v^i\right)\left(\delta\rho\delta v^k\right) - \nabla_k\left(\rho \delta v^i \delta v^k\right) \\ &- \nabla^i \left(\frac{1}{2} \frac{\mathrm{d}^2 p}{\mathrm{d}\rho^2} \delta\rho^2 \right) - \frac{1}{8\pi} \nabla^i \left(\delta B^k \delta B_k \right) - \frac{1}{4\pi} \nabla_k \left(\delta B^i \delta B^k \right) \\ &- \nabla_k \left(\delta\rho \delta v^i \delta v^k \right) + \mathcal{O}\left(\delta\rho^3\right) \end{aligned}$$

and $c_{\mathrm{s}} = (\mathrm{d}p/\mathrm{d}\rho)^{1/2}$ denotes the sound speed.

Using the continuity equation to eliminate the density perturbation from the left hand side of the Euler equation, we arrive at a single nonlinear equation governing the velocity perturbation

$$\hat{L}\, \delta \boldsymbol{v} = \boldsymbol{\mathcal{N}} \,. \tag{9}$$

The linear differential operator $\hat{L}$ is defined as

$$\hat{L}\, \delta v^i = \rho \left[\frac{\partial \delta v^i}{\partial t} + v^k \nabla_k \delta v^i + \left(\nabla_k v^i\right) \delta v^k \right] - \rho \nabla^i \left[\frac{c_{\mathrm{s}}^2}{\rho} \partial_\tau^{-1} \nabla_k \left(\rho \delta v^k\right) \right] \tag{10}$$

and the nonlinear part is given by

$$\mathcal{N}^i = \mathcal{N}_v^i - \rho \nabla^i \left[\frac{c_{\mathrm{s}}^2}{\rho} \partial_\tau^{-1} \mathcal{N}_\rho \right] , \tag{11}$$

where ∂_τ^{-1} is the inverse operator to $\partial_\tau = \partial/\partial t + \Omega\, \partial/\partial\phi$ (note that in the space of the quadratically integrable functions this inversion makes sense).

If the nonlinearities are neglected, (i.e. when $\mathcal{N}^i = 0$), the Eq. (9) describes propagation of the acoustic or inertial waves on the stationary laminar background. If, in addition, a suitable boundary conditions are specified, this equation gives us the set of the normal modes of the system. In the presence of a *weak* turbulence we assume that both, the oscillations and turbulent fluctuations, can be treated as perturbation to the background stationary flow. Hence, we decompose the perturbation of any quantity $q = \{\rho, v^i, B^i\}$ into the part due to the oscillations and the one due to turbulence,

$$\delta q = \delta q_{\mathrm{osc}} + \delta q_{\mathrm{turb}} \,. \tag{12}$$

and assume a regime when

$$|\delta q_{\mathrm{osc}}| \ll |\delta q_{\mathrm{turb}}| \ll |q| \,. \tag{13}$$

Therefore, we safely neglect any influence of the oscillations on the turbulence, but we suppose that properties of both, the background flow and the turbulence, affect the oscillations.

If we now introduce the Lagrangian displacement ξ^i corresponding to the oscillations using the relations,

$$\delta v^i_{\rm osc} = \frac{\partial \xi^i}{\partial t} + v^k \nabla_k \xi^i - \xi^j \nabla_j v^i\,, \quad \delta\rho_{\rm osc} = -\nabla_k\big(\rho\xi^k\big)\,, \tag{14}$$

and substitute into the Eq. (9) keeping only terms linear in ξ^i, we arrive at the master equation

$$\left(\hat{\mathcal{L}}^i_j + \hat{\mathcal{D}}^i_j\right)\xi^j = \mathcal{S}^i\,. \tag{15}$$

The operator $\hat{\mathcal{L}}$ is defined by

$$\hat{\mathcal{L}}^i_j = \delta^i_j\left(\frac{\partial}{\partial t} + v^k\nabla_k\right)^2 - \frac{1}{\rho}\Big[(\gamma-1)\nabla^i(p\nabla_j) + \nabla_j\big(p\nabla^i\big)\Big] + \nabla^i\nabla_j\Phi\,. \tag{16}$$

and is deterministic being given solely by the background flow quantities. On the other hand, $\hat{\mathcal{D}}^i_j$ and $\mathcal{S}^i$ are stochastic because they rely on both the background flow and the turbulent field. The operator $\hat{\mathcal{D}}^i_j$ is a contribution of nonlinearity $\mathcal{N}^i$; the part which is linear in $\delta v_{\rm osc}$ and $\delta\rho_{\rm osc}$, and therefore in ξ^i as well. It modify the operator $\hat{\mathcal{L}}^i_j$ and therefore slightly change the eigenfunctions and eigenfrequencies of the oscillation modes. The term $\mathcal{S}^i$ on the right-hand side depends only on the turbulent fluctuations and play the role of the stochastic source term. We do not show complicated expressions of these operator as the they are not needed in the rest of the paper. For a general discussion presented here their stochastic nature and the structure of Eq. (15) are sufficient.

3 DECOMPOSITION INTO NORMAL MODES

In absence of the turbulence, both $\hat{\mathcal{L}}$ and $\mathcal{S}$ vanish and the Eq. (15) takes the form

$$\hat{\mathcal{L}}\boldsymbol{\xi} = \partial_t^2\boldsymbol{\xi} + \hat{B}\,\partial_t\boldsymbol{\xi} + \hat{C}\boldsymbol{\xi} = 0\,, \tag{17}$$

where $\hat{B}$ and $\hat{C}$ are two linear differential operators. With appropriate boundary conditions, this equation describes linear modes of the system. Glampedakis and Andersson (2007) show that in absence of electromagnetic radiation on the surface of the body, the operator $\hat{C}$ is Hermitian and $\hat{B}$ is anti-Hermitian with respect to the standard scalar product weighted by mass density,

$$\langle\boldsymbol{\zeta},\boldsymbol{\eta}\rangle = \int_V (\boldsymbol{\zeta}^*\cdot\boldsymbol{\eta})\,\rho\,{\rm d}V\,. \tag{18}$$

Assuming a harmonic time dependence for the perturbation, i.e. $\boldsymbol{\xi} = \boldsymbol{\zeta}(\boldsymbol{x})\exp[-{\rm i}wt]$, we obtain a set of linear modes $\{\omega_\alpha, \boldsymbol{\zeta}_\alpha\}$, each of them characterized by its eigenfrequency ω_α and eigenfunction $\boldsymbol{\zeta}_\alpha$. As shown by Schenk et al. (2002), the eigenfunctions (completed by the associated functions in the case of Jordan-chain modes) can be used as a basis of the corresponding phase space $\mathcal{H} \oplus \mathcal{H}$. The solution of the general equation

$$\hat{\mathcal{L}}\boldsymbol{\xi} = \boldsymbol{\mathcal{F}}(t, \boldsymbol{x}) \tag{19}$$

can be expressed as a linear combination of modal eigenfunctions

$$\begin{pmatrix} \boldsymbol{\xi} \\ \partial_t \boldsymbol{\xi} \end{pmatrix} = \sum_\alpha c_\alpha(t) \begin{pmatrix} \boldsymbol{\zeta}_\alpha \\ -\mathrm{i}\omega_\alpha \boldsymbol{\zeta}_\alpha \end{pmatrix}, \tag{20}$$

in which the coefficients $c_\alpha(t)$ satisfy the equations of forced oscillators

$$\frac{\mathrm{d}c_\alpha}{\mathrm{d}t} + \mathrm{i}\omega_\alpha c_\alpha = -\frac{\mathrm{i}}{b_\alpha}\left\langle \boldsymbol{\zeta}_\alpha, \boldsymbol{\mathcal{F}} \right\rangle \tag{21}$$

and $b_\alpha = \langle \boldsymbol{\zeta}_\alpha, \mathrm{i}\hat{B}\boldsymbol{\zeta}_\alpha \rangle + 2\omega_\alpha \langle \boldsymbol{\zeta}_\alpha, \boldsymbol{\zeta}_\alpha \rangle$.

Equation (15) represents a special case for which $\boldsymbol{\mathcal{F}} = -\hat{\mathcal{D}}\boldsymbol{\xi} + \boldsymbol{\mathcal{S}}$. With the aid of expansion (20), the Eq. (21) becomes

$$\frac{\mathrm{d}c_\alpha}{\mathrm{d}t} + \mathrm{i}\sum_\beta \left(\omega_\alpha \delta_{\alpha\beta} - \epsilon \mathcal{D}_{\alpha\beta}\right) c_\beta = \epsilon \mathcal{S}_\alpha \,, \tag{22}$$

where

$$\epsilon \mathcal{D}_{\alpha\beta} = \frac{1}{b_\alpha}\left\langle \boldsymbol{\zeta}_\alpha, \hat{\mathcal{D}}\boldsymbol{\zeta}_\beta \right\rangle, \quad \epsilon \mathcal{S}_\alpha = \frac{1}{b_\alpha}\left\langle \boldsymbol{\zeta}_\alpha, \boldsymbol{\mathcal{S}} \right\rangle . \tag{23}$$

For subsonic turbulence, these terms are small and can be treated as perturbations. That is why we formally introduced a small parameter $\epsilon \ll 1$ to the Eq. (22).

4 COHERENT AND RANDOM OSCILLATIONS

The stochastic functions $\mathcal{D}_{\alpha\beta}(t)$ and $S_\alpha(t)$ can be separated to the mean and fluctuating random components,

$$\mathcal{D}_{\alpha\beta}(t) = \bar{D}_{\alpha\beta} + D'_{\alpha\beta}(t) \,, \quad \mathcal{S}_\alpha(t) = \bar{S}_\alpha + S'_\alpha(t) \,. \tag{24}$$

The mean values and all statistical moments of the random components are assumed to be time-independent as a consequence of a stationary turbulence. Similarly, we suppose that also the resulting oscillations can be separated to coherent and random components as $c_\alpha(t) = \bar{c}_\alpha(t) + c'_\alpha(t)$. Ansamble average of Eq. (22) gives

$$\frac{\mathrm{d}\bar{c}_\alpha}{\mathrm{d}t} + \mathrm{i}\omega_\alpha \bar{c}_\alpha - \mathrm{i}\epsilon \sum_\beta \left[\bar{\mathcal{D}}_{\alpha\beta}\bar{c}_\beta + \left\langle \mathcal{D}'_{\alpha\beta} c'_\beta \right\rangle \right] = \epsilon \bar{S}_\alpha \,. \tag{25}$$

Subtracting it from the original Eq. (22) we obtain the equation governing the random component

$$\frac{\mathrm{d}c'_\alpha}{\mathrm{d}t} + \mathrm{i}w_\alpha c'_\alpha - \mathrm{i}\epsilon \sum_\beta \left[\bar{\mathcal{D}}_{\alpha\beta} c'_\beta + \mathcal{D}'_{\alpha\beta}\bar{c}_\beta + \mathcal{D}'_{\alpha\beta} c'_\beta - \left\langle \mathcal{D}'_{\alpha\beta} c'_\beta \right\rangle \right] = \epsilon \mathcal{S}'_\alpha \,. \tag{26}$$

We solve these equations using method of multiple time-scales. Instead of the physical time t, we introduce variables $T_n = \epsilon^n t$ with $n = 0, 1, 2, \ldots$ The time derivative is then replaced by the series

$$\frac{\mathrm{d}}{\mathrm{d}t} = \partial_0 + \epsilon\partial_1 + \epsilon^2\partial_2 + \ldots, \quad \partial_n = \frac{\partial}{\partial T_n} \tag{27}$$

and the solutions are looked for in the form

$$\bar{c}_\alpha(T_n) = \bar{c}_\alpha^{(0)}(T_n) + \epsilon\bar{c}_\alpha^{(1)}(T_n) + \epsilon^2\bar{c}_\alpha^{(2)}(T_n) + \ldots, \tag{28}$$

$$c'_\alpha(T_n) = c_\alpha'^{(0)}(T_n) + \epsilon c_\alpha'^{(1)}(T_n) + \epsilon^2 c_\alpha'^{(2)}(T_n) + \ldots \tag{29}$$

The zero-order equations,

$$(\partial_0 + \mathrm{i}\omega_\alpha)\,\bar{c}_\alpha^{(0)} = 0, \quad (\partial_0 + \mathrm{i}\omega_\alpha)\,c_\alpha'^{(0)} = 0, \tag{30}$$

give solutions

$$\bar{c}_\alpha^{(0)} = A_\alpha(T_1, T_2, \ldots)\,\mathrm{e}^{-\mathrm{i}\omega_\alpha T_0}, \quad c_\alpha'^{(0)} = 0. \tag{31}$$

The fluctuating component vanishes as a consequence of the assumption that $\langle c'_\alpha \rangle = 0$.

The first-order equations read

$$\begin{aligned}(\partial_0 + \mathrm{i}\omega_\alpha)\,\bar{c}_\alpha^{(1)} &= -\,\partial_1\bar{c}_\alpha^{(0)} + \mathrm{i}\sum_\beta \bar{\mathcal{D}}_{\alpha\beta}\bar{c}_\beta^{(0)} + \bar{\mathcal{S}}_\alpha = \\ &\quad -\,(\partial_1 A_\alpha)\,\mathrm{e}^{-\mathrm{i}\omega_\alpha T_0} + \mathrm{i}\sum_\beta \bar{\mathcal{D}}_{\alpha\beta}A_\beta\mathrm{e}^{-\mathrm{i}\omega_\beta T_0} + \bar{\mathcal{S}}_\alpha, \\ (\partial_0 + \mathrm{i}\omega_\alpha)\,c_\alpha'^{(1)} &= \mathrm{i}\sum_\beta \mathcal{D}'_{\alpha\beta}\bar{c}_\beta^{(0)} + \mathcal{S}'_\alpha.\end{aligned} \tag{32}$$

For simplicity, we suppose that there is no degeneracy (i.e. if $\alpha \neq \beta$ then also $\omega_\alpha \neq \omega_\beta$). The secular terms on the right hand side of Eq. (4) are therefore eliminated when

$$-\partial_1 A_\alpha + \mathrm{i}\bar{\mathcal{D}}_{\alpha\alpha}A_\alpha = 0 \tag{33}$$

the solution of which is

$$A_\alpha(T_1, T_2, \ldots) = A_\alpha(T_2, \ldots)\exp\left[-\mathrm{i}\omega_\alpha^{(1)}T_1\right], \quad \omega_\alpha^{(1)} = -\bar{\mathcal{D}}_{\alpha\alpha}. \tag{34}$$

Hence, $-\epsilon\bar{\mathcal{D}}_{\alpha\alpha}$ is a first-order correction to the eigenfrequency of the coherent oscillations. A particular solution for the coherent part is

$$\bar{c}_\alpha^{(1)} = -\frac{\mathrm{i}}{\omega_\alpha}\bar{\mathcal{S}}_\alpha + \sum_{\beta\neq\alpha}\frac{\bar{\mathcal{D}}_{\alpha\beta}}{\omega_\alpha - \omega_\beta}A_\beta\mathrm{e}^{-\mathrm{i}\omega_\beta T_0} \tag{35}$$

and for the random part

$$c_\alpha'^{(1)} = \mathrm{e}^{-\mathrm{i}\omega_\alpha T_0}\left(\sum_\beta I_{\alpha\beta}A_\beta + J_\alpha\right), \tag{36}$$

where

$$I_{\alpha\beta} = \int_{-\infty}^{T_0} \mathcal{D}'_{\alpha\beta}(\tau) \mathrm{e}^{\mathrm{i}(\omega_\alpha - \omega_\beta)\tau}\, \mathrm{d}\tau\,, \quad J_\alpha = \int_{-\infty}^{T_0} \mathcal{S}'_\alpha(\tau) \mathrm{e}^{\mathrm{i}\omega_\alpha \tau}\, \mathrm{d}\tau. \tag{37}$$

A second-order approximation for the coherent part of oscillations is governed by

$$\begin{aligned}(\partial_0 + \mathrm{i}\omega_\alpha)\, \bar{c}_\alpha^{(2)} &= -\, \partial_2 \bar{c}_\alpha^{(0)} - \partial_1 \bar{c}_\alpha^{(1)} + \mathrm{i} \sum_\beta \left(\bar{\mathcal{D}}_{\alpha\beta} \bar{c}_\beta^{(1)} + \left\langle \mathcal{D}'_{\alpha\beta} c_\beta'^{(1)} \right\rangle \right) = \\ &= -\, (\partial_2 A_\alpha)\, \mathrm{e}^{-\mathrm{i}\omega_\alpha T_0} - \sum_{\beta \neq \alpha} \frac{\bar{\mathcal{D}}_{\alpha\beta} \bar{\mathcal{D}}_{\beta\beta}}{\omega_\alpha - \omega_\beta} A_\beta \mathrm{e}^{-\mathrm{i}\omega_\beta T_0} \\ &\quad + \sum_\beta \frac{1}{\omega_\beta} \bar{\mathcal{D}}_{\alpha\beta} \bar{\mathcal{S}}_\beta + \mathrm{i} \sum_\beta \sum_{\gamma \neq \beta} \frac{\bar{\mathcal{D}}_{\alpha\beta} \bar{\mathcal{D}}_{\beta\gamma}}{\omega_\beta - \omega_\gamma} A_\gamma \mathrm{e}^{\mathrm{i}\omega_\gamma T_0} \\ &\quad + \mathrm{i} \sum_\beta \left[\sum_\gamma \left\langle \mathcal{D}'_{\alpha\beta} I_{\beta\gamma} \right\rangle A_\gamma + \left\langle \mathcal{D}'_{\alpha\beta} J_\beta \right\rangle \right] \mathrm{e}^{-\mathrm{i}\omega_\beta T_0}\,. \end{aligned} \tag{38}$$

Due to the stationarity, the correlators on the right-hand side can be expressed as

$$\left\langle \mathcal{D}'_{\alpha\beta} I_{\beta\gamma} \right\rangle = C_{\alpha\beta\beta\gamma} \exp\left[\mathrm{i}(\omega_\beta - \omega_\gamma) T_0 \right]\,, \tag{39}$$

$$\left\langle \mathcal{D}'_{\alpha\beta} J_\beta \right\rangle = C_{\alpha\beta\beta} \exp\left[\mathrm{i}\omega_\beta T_0 \right]\,, \tag{40}$$

where

$$C_{\alpha\beta\beta\gamma} = \int_{-\infty}^{0} \left\langle \mathcal{D}'_{\alpha\beta}(0) \mathcal{D}'_{\beta\gamma}(\tau) \right\rangle \mathrm{e}^{\mathrm{i}(\omega_\beta - \omega_\gamma)\tau} \mathrm{d}\tau\,, \tag{41}$$

$$C_{\alpha\beta\beta} = \int_{-\infty}^{0} \left\langle \mathcal{D}'_{\alpha\beta}(0) \mathcal{S}'_\beta(\tau) \right\rangle \mathrm{e}^{\mathrm{i}\omega_\beta \tau} \mathrm{d}\tau \tag{42}$$

are constants. The secular terms on the right-hand side of Eq. (38) are eliminated when

$$-\partial_2 A_\alpha + \mathrm{i} \sum_{\beta \neq \alpha} \frac{\bar{\mathcal{D}}_{\alpha\beta} \bar{\mathcal{D}}_{\beta\alpha}}{\omega_\beta - \omega_\alpha} A_\alpha + \mathrm{i} \sum_\beta C_{\alpha\beta\beta\alpha} A_\alpha = 0\,, \tag{43}$$

the solution of which is

$$A(T_2) = A \exp\left[-\mathrm{i}\omega_\alpha^{(2)} T_2 \right], \quad \omega_\alpha^{(2)} = -\sum_{\beta \neq \alpha} \frac{\bar{\mathcal{D}}_{\alpha\beta} \bar{\mathcal{D}}_{\beta\alpha}}{\omega_\beta - \omega_\alpha} - \sum_\beta C_{\alpha\beta\beta\alpha}\,. \tag{44}$$

Therefore the eigenfrequency of the coherent oscillations in the turbulent flow is shifted with respect to the eigenfrequency of the background laminar flow by a correction

$$\Delta\omega_\alpha = -\epsilon \bar{\mathcal{D}}_{\alpha\alpha} - \epsilon^2 \sum_{\beta \neq \alpha} \frac{\bar{\mathcal{D}}_{\alpha\beta} \bar{\mathcal{D}}_{\beta\alpha}}{\omega_\beta - \omega_\alpha} - \epsilon^2 \sum_\beta C_{\alpha\beta\beta\alpha}\,. \tag{45}$$

5 DISCUSSION AND CONCLUSIONS

In this note, effects of turbulence on oscillations of rotating flow have been examined. Starting from the ideal MHD equations, we derived IWE describing excitation and damping of the oscillations. Using decomposition into normal modes we obtained general formulae for amplitudes of random oscillations [equation (36)] and for the frequency shifts of the coherent oscillations [equation (45)]. If they are imaginary, these frequency shifts may describe damping or instabilities induced by the turbulence. Further exploration of this issue needs a specific model of the turbulence and is left for a further work.

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