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The gyraton solutions on generalized Melvin universe with cosmological constant

Hedvika Kadlecová[a] and Pavel Krtouš[b]

Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University,
V Holešovičkách 2, 180 00 Prague, Czech Republic.
[a]hedvika.kadlecova@centrum.cz
[b]Pavel.Krtous@utf.mff.cuni.cz

ABSTRACT
We present and analyse new exact gyraton solutions of algebraic type II on generalized Melvin universe of type D which admit non-vanishing cosmological constant Λ. We show that it generalizes both, gyraton solutions on Melvin and on direct product spacetimes. When we set $\Lambda = 0$ we get solutions on Melvin spacetime and for $\Sigma = 1$ we obtain solutions on direct product spacetimes. We demonstrate that the solutions are member of the Kundt family of spacetimes as its subcases. We show that the Einstein equations reduce to a set of equations on the transverse 2-space. We also discuss the polynomial scalar invariants which are non-constant in general but constant for sub-solutions on direct product spacetimes.

Keywords: Gyraton solutions – Melvin universe – cosmological constant – Kundt family – direct product spacetimes – constant polynomial scalar invariants – Einstein equations

1 INTRODUCTION

In Kadlecová et al. (2009) and Kadlecová and Krtouš (2010) we have investigated the gyraton solutions on direct product spacetimes and gyraton solutions on Melvin universe. These solutions are of algebraic type II. In this work we present the gyraton solutions on Melvin universe with the cosmological constant.

We present our ansatz for the gyraton metric on generalized Melvin universe and the generalized electromagnetic tensor. We briefly review the derivation of the Einstein–Maxwell equations. The source-free Einstein equations determine the functions Σ and S, in particular, there exists a relation between them. Next we derive the non-trivial source equations. The Einstein–Maxwell equations do decouple for the gyraton metric on generalized Melvin universe as for its subcase solutions on Melvin and on direct product spacetimes. Next, we focus on interpretation of our solutions. Especially, we discuss the geometry of the transverse metric of the generalized Melvin universe in detail for different values of the cosmological constant. We show explicitly that the Melvin universe and direct product spacetimes are special cases of our solutions. We also discuss the properties of the scalar polynomial invariants which are functions of ρ but for subcase solutions on direct product spacetimes ($\Sigma = 1$) the invariants are constant.

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2 THE ANSATZ FOR THE GYRATONS ON GENERALIZED MELVIN UNIVERSE

The ansatz for the gyraton metric on the generalized Melvin spacetime is the following,

$$\mathbf{g} = -2\Sigma^2 H\, \mathrm{d}u^2 - \Sigma^2\, \mathrm{d}u \vee \mathrm{d}v + \mathbf{q} + \Sigma^2 \mathrm{d}u \vee \mathbf{a}\,, \tag{1}$$

where we have introduced the 2-dimensional transversal metric $\mathbf{q}$ on transverse spaces $u,\ v = \text{constant}$ as

$$\mathbf{q} = \Sigma^2\, \mathrm{d}\rho^2 + \frac{S(\rho)^2}{\Sigma^2}\, \mathrm{d}\phi^2\,. \tag{2}$$

We have assumed that the metric (1) belongs to the Kundt class of spacetimes and that the transversal metric $\mathbf{q}$ has one Killing vector $\mathcal{L}_{\partial/\partial\phi}\mathbf{q} = 0$. The metric (1) represents gyraton propagating on the background which is formed by generalized Melvin spacetime. The metric (1) generalizes only the transversal metric therefore the algebraical type is II as for the gyraton on the Melvin spacetime Kadlecová and Krtouš (2010), the NP quantities are listed in Kadlecová (2013).

We have generalized the transversal metric for the Melvin universe by assuming general function $S = S(\rho)$ instead of the simple coordinate ρ in front of the term $\mathrm{d}\phi^2$, see Kadlecová and Krtouš (2010). We will show that these general functions $\Sigma(\rho)$ and $S(\rho)$ are determined by the Einstein–Maxwell equations and have proper interpretation. The presence of cosmological constant Λ is not allowed for the solution on pure Melvin background Kadlecová and Krtouš (2010).

The transverse space is covered by two spatial coordinates x^i $(i = \rho, \phi)$ and it is convenient to introduce suitable notation on it, technical details can be found in Kadlecová (2013). The function $H(u, v, \mathbf{x})$ in the metric (1) can depend on all coordinates, but the functions $a(u, \mathbf{x})$ are v-independent.

The derivation of the Einstein–Maxwell equations is almost identical with the previous paper Kadlecová and Krtouš (2010) therefore we will describe the derivation of Einstein–Maxwell equations very briefly.

The metric should satisfy the Einstein equations with cosmological constant Λ and with a stress-energy tensor generated by the electromagnetic field of the background Melvin spacetime $\mathbf{T}^{\mathrm{EM}}$ and the gyratonic source $\mathbf{T}^{\mathrm{gyr}}$ as[1]

$$\mathbf{G} + \Lambda\, \mathbf{g} = \varkappa\big(\mathbf{T}^{\mathrm{EM}} + \mathbf{T}^{\mathrm{gyr}}\big)\,. \tag{3}$$

We assume the electromagnetic field is given by

$$\mathbf{F} = E\, \mathrm{d}v \wedge \mathrm{d}u + \frac{B}{\Sigma^2}\boldsymbol{\epsilon} + \mathrm{d}u \wedge \big(E\, \mathbf{s} - B{*}(\mathbf{s} - \mathbf{a})\big)\,, \tag{4}$$

[1] $\varkappa = 8\pi G$ and ε_0 are gravitational and electromagnetic constants. There are two general choices of geometrical units: the gaussian with $\varkappa = 8\pi$ and $\varepsilon_0 = 1/4\pi$, and SI-like with $\varkappa = \varepsilon_0 = 1$.

where E and B are parameters of electromagnetic field. The self-dual complex form of the Maxwell[2] tensor is

$$\mathcal{F} = \mathcal{B}\left(\mathrm{d}v \wedge \mathrm{d}u - \frac{i}{\Sigma^2}\boldsymbol{\epsilon} + \mathrm{d}u \wedge \left[\mathbf{s} + i{*}(\mathbf{s} - \boldsymbol{a})\right]\right), \tag{5}$$

for details see Kadlecová and Krtouš (2010).

We have denoted the complex constant $\mathcal{B} = E + iB$, and we have introduced a constant ϱ_{EM},

$$\varrho_{\mathrm{EM}} = \frac{\varkappa \varepsilon_{\mathrm{o}}}{2}\left(E^2 + B^2\right). \tag{6}$$

We define the gyratonic matter only on a phenomenological level as

$$\varkappa\, \mathbf{T}^{\mathrm{gyr}} = j_u\, \mathrm{d}u^2 + \mathrm{d}u \vee \mathbf{j}\,, \tag{7}$$

where the source functions $j_u(v, u, \mathbf{x})$ and $j(v, u, \mathbf{x})$. We assume that the gyraton stress-energy tensor is locally conserved,

$$\nabla \cdot \mathbf{T}^{\mathrm{gyr}} = 0\,. \tag{8}$$

To conclude, the fields are characterized by functions Σ, S, H, $\mathbf{a}$, and $\mathbf{s}$ which must be determined by the field equations and the gyraton sources j_u and $\mathbf{j}$ and the constants E and B of the background electromagnetic field are prescribed.

3 THE EINSTEIN–MAXWELL FIELD EQUATIONS

First, we will start to solve the Maxwell equations, it is sufficient to calculate the cyclic Maxwell equation for the self-dual Maxwell tensor (5)

$$0 = \mathrm{d}\mathcal{F} = \mathcal{B}\left\{\partial_v\big(\mathbf{s} + i{*}(\mathbf{s} - \mathbf{a})\big)\,\mathrm{d}v \wedge \mathrm{d}u \wedge \mathrm{d}\mathbf{x} - \left[\operatorname{rot}\mathbf{s} + i\operatorname{div}(\mathbf{s} - \mathbf{a})\right]\mathrm{d}u \wedge \boldsymbol{\epsilon}\right\}. \tag{9}$$

From the real part we immediately get that the 1-forms $\mathbf{s}$ is v-independent, and rotation free $\operatorname{rot}\mathbf{s} = 0$. From imaginary part it follows that the 1-form $\mathbf{a}$ is also independent and it satisfies $\operatorname{div}(\mathbf{s} - \mathbf{a}) = 0$.

3.1 The trivial Einstein–Maxwell equations – determining the function Σ and S

Next we will derive the Einstein–Maxwell equations from the Einstein tensor and the electromagnetic stress-energy tensor, which are listed in Kadlecová (2013).

First we will solve the equations which are source free and we will be able to determine the analytic formula for the functions Σ and S.

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[2] We will follow the notation of Stephani et al. (2003). Namely, $\mathcal{F} \equiv \mathbf{F} + i{\star}\mathbf{F}$ is complex self-dual Maxwell tensor, where the 4-dimensional Hodge dual is ${\star}F_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}/2$. The self-dual condition reads ${\star}\mathcal{F} = -i\mathcal{F}$. The orientation of the 4-dimensional Levi–Civita tensor is fixed by the sign of the component $\varepsilon_{vu\rho\phi} = S\Sigma^2$. The energy-momentum tensor of the electromagnetic field is given by $T_{\mu\nu} = \varepsilon_{\mathrm{o}}\mathcal{F}_{\mu}^{\ \rho}\overline{\mathcal{F}}_{\nu\rho}/2$.

The first equation we obtain from the vu-component,

$$-\frac{(\Sigma_{,\rho})^2}{\Sigma^2}+2\frac{\Sigma_{,\rho}}{\Sigma}\frac{S_{,\rho}}{S}-\frac{S_{,\rho\rho}}{S}=\Lambda\Sigma^2+\frac{\varrho_{\mathrm{EM}}}{\Sigma^2}\,, \tag{10}$$

the next two equations we get from the transverse diagonal components $\rho\rho$ and $\phi\phi$,

$$-\frac{(\Sigma_{,\rho})^2}{\Sigma^2}+2\frac{\Sigma_{,\rho}}{\Sigma}\frac{S_{,\rho}}{S}+\partial_v^2 H=-\Lambda\Sigma^2+\frac{\varrho_{\mathrm{EM}}}{\Sigma^2}\,, \tag{11}$$

$$-\frac{(\Sigma_{,\rho})^2}{\Sigma^2}+2\frac{\Sigma_{,\rho\rho}}{\Sigma}+\partial_v^2 H=-\Lambda\Sigma^2+\frac{\varrho_{\mathrm{EM}}}{\Sigma^2}\,. \tag{12}$$

When we compare the equation (11) and (12) we immediately get the relation between the functions Σ and S, as $\Sigma_{,\rho}S_{,\rho}/S=\Sigma_{,\rho\rho}$, and thus we are able to determine their explicit relation ($\Sigma_{,\rho}\neq 0$) as

$$\Sigma_{,\rho}=\gamma S\,, \tag{13}$$

where γ is an integration constant.

After substituting the relation (13) into Eq. (10) then we get equation

$$-\frac{(\Sigma_{,\rho})^2}{\Sigma^2}+2\frac{\Sigma_{,\rho\rho}}{\Sigma}+\frac{\Sigma_{,\rho\rho\rho}}{\Sigma_{,\rho}}=\Lambda\Sigma^2+\frac{\varrho_{\mathrm{EM}}}{\Sigma^2}\,, \tag{14}$$

which will be useful later.

To determine the function H it is useful to substitute (13) into the Eq. (12) and then multiply it by $\Sigma/2\Sigma_{,\rho}$, we get

$$\frac{1}{2}\big(\partial_v^2 H\big)_{,\rho}\frac{\Sigma}{\Sigma_{,\rho}}-2\frac{\Sigma_{,\rho\rho}}{\Sigma}+\frac{\big(\Sigma_{,\rho}\big)^2}{\Sigma^2}+\frac{\Sigma_{,\rho\rho\rho}}{\Sigma_{,\rho}}=-\Lambda\Sigma^2-\frac{\varrho_{\mathrm{EM}}}{\Sigma^2}\,. \tag{15}$$

Now, we add the Eq. (10) to (15) and obtain, $(\partial_v^2 H)_{,\rho}\Sigma/2\Sigma_{,\rho}=0$, then for $\Sigma_{,\rho}\neq 0$ we can write that $\partial_v^2 H=-\alpha$, where α is a constant.

Thus the metric function H has a structure

$$H=-\frac{1}{2}\alpha v^2+g\,v+h\,, \tag{16}$$

where we have introduced v-independent functions $g(u,\mathbf{x})$ and $h(u,\mathbf{x})$.

In the following we want to determine an analytical expression for Σ, in order to do that we substitute the result (16) into (12),

$$2\frac{\Sigma_{,\rho\rho}}{\Sigma}-\frac{\big(\Sigma_{,\rho}\big)^2}{\Sigma^2}=-\Lambda\Sigma^2+\frac{\varrho_{\mathrm{EM}}}{\Sigma^2}+\alpha\,. \tag{17}$$

When we add the expression (14) to (17), we obtain that

$$\Sigma_{,\rho\rho\rho}=-2\Lambda\Sigma^2\Sigma_{,\rho}+\alpha\Sigma_{,\rho}\,. \tag{18}$$

We can rewrite the previous equation as $\Sigma_{,\rho\rho\rho} = -2\Lambda(\Sigma^3)_{,\rho}/3 + \alpha\Sigma_{,\rho}$ to be able to integrate it again as

$$\Sigma_{,\rho\rho} = -\frac{2}{3}\Lambda\Sigma^3 + \alpha\Sigma + \frac{1}{2}\beta\,, \tag{19}$$

which we can rewrite as

$$\frac{1}{2}\big[(\Sigma_{,\rho})^2\big]_{,\rho} = -\frac{1}{6}\Lambda\big(\Sigma^4\big)_{,\rho} + \alpha\big(\Sigma^2\big)_{,\rho} + \frac{1}{2}\beta\Sigma_{,\rho}\,. \tag{20}$$

After another integration we get the final formula for the derivative of the function Σ,

$$\big(\Sigma_{,\rho}\big)^2 = -\frac{1}{3}\Lambda\Sigma^4 + \alpha\Sigma^2 + \beta\Sigma + c\,, \tag{21}$$

and it can be rewritten using (13) as

$$\gamma S = \left[-\frac{1}{3}\Lambda\Sigma^4 + \alpha\Sigma^2 + \beta\Sigma + c\right]^{1/2}\,, \tag{22}$$

where α, β and c are integration constants which should be determined.

Furthermore, we are able to determine the constant c explicitly. When we substitute the result (21) and (19) into (17) we immediately obtain that $c = -\varrho_{\mathrm{EM}}$. The constants α and β will be determined in the Section 4.1.

3.2 The Einstein–Maxwell equations for the sources

The remaining nontrivial components of the Einstein equations are those involving the gyraton source (7). To write the source equation we have to evaluate the component G_{uv} using the expressions for derivatives of Σ. Then the component G_{uv} has the explicit form

$$G_{uv} = \Lambda\Sigma^2 + \frac{\varrho_{\mathrm{EM}}}{\Sigma^2}\,. \tag{23}$$

The ui-components give equations related to $\mathbf{j}$,

$$\Sigma^2\,\mathbf{j} = \frac{1}{2}\mathrm{rot}\left(\Sigma^4\, b\right) + \Sigma^2\mathrm{d}g - \alpha\Sigma^2\mathbf{a} + 2\varrho_{\mathrm{EM}}(\mathbf{s} - \mathbf{a})\,, \tag{24}$$

where $b = \mathrm{rot}\ \mathbf{a}$.

It is useful to split the source equation into divergence and rotation parts:

$$\mathrm{div}\left(\Sigma^2\,\mathbf{j}\right) = \mathrm{div}\ \Sigma^2(\mathrm{d}g - \alpha\,\mathbf{a})\,, \tag{25}$$

$$\mathrm{rot}\left(\Sigma^2\mathbf{j}\right) = -\frac{1}{2}\triangle\left(\Sigma^4 b\right) + \mathrm{rot}\left(\Sigma^2\mathrm{d}g\right) - \alpha\mathrm{rot}\left(\Sigma^2\mathbf{a}\right) - 2\varrho_{\mathrm{EM}}\, b\,. \tag{26}$$

These are coupled equations for g and $\mathbf{a}$. We will return to them below.

The condition (8) for the gyraton source gives, that the sources $\mathbf{j}$ must be v-independent and j_u has the structure

$$j_u = v\ \mathrm{div}\big(\Sigma^2\mathbf{j}\big) + \iota\,, \tag{27}$$

where $\iota(u, \mathbf{x})$ is v-independent function, see Kadlecová and Krtouš (2010) Eq. (2.51). The gyraton source (7) is therefore determined by three v-independent functions $\iota(u, \mathbf{x})$ and $j(u, \mathbf{x})$.

The uu-component of the Einstein equation gives

$$j_u = v\Big[\operatorname{div}\big(\Sigma^2 \mathrm{d}g\big) - \alpha \operatorname{div}\big(\Sigma^2 \mathbf{a}\big)\Big] + \Sigma^2\Big(\triangle h - \big(\Sigma^{-2}\big)_{,\rho} h_{,\rho}\Big) \\ + \frac{1}{2}\Sigma^4 b^2 + 2\Sigma^2 \mathbf{a}\cdot \mathrm{d}g + (\partial_u + g)\operatorname{div}\big(\Sigma^2\mathbf{a}\big) - \alpha\Sigma^2\mathbf{a}^2 - 2\varrho_{\mathrm{EM}}\,(\mathbf{s}-\mathbf{a})^2 . \quad (28)$$

Then we can compare the coefficient in front of v with (25) and we get consistent structure with (27). The nontrivial v-independent part of (28) gives the equation for the metric function h,

$$\Sigma^2\Big(\triangle h - \big(\Sigma^{-2}\big)_{,\rho} h_{,\rho}\Big) = \iota - \frac{1}{2}\Sigma^4 b^2 - 2\Sigma^2\mathbf{a}\cdot\mathrm{d}g \\ - (\partial_u + g)\operatorname{div}\big(\Sigma^2\mathbf{a}\big) + \alpha\Sigma^2\mathbf{a}^2 + 2\varrho_{\mathrm{EM}}\big(\mathbf{s}-\mathbf{a}\big)^2 . \quad (29)$$

Now, let us return to solution of Eqs. (25) and (26). The first equation simplifies if we use gauge condition

$$\operatorname{div}\big(\Sigma^2\mathbf{a}\big) = 0 . \quad (30)$$

It can be satisfied due to gauge freedom $v \to v - \chi$, $\mathbf{a} \to \mathbf{a} - \mathrm{d}\chi$, cf. the discussion in Kadlecová and Krtouš (2010). Such a condition implies the existence of a potential $\tilde{\lambda}$, as $\Sigma^2\,\mathbf{a} = \operatorname{rot}\tilde{\lambda}$.

The equation (25) now reduces to

$$\operatorname{div}\big(\Sigma^2\mathrm{d}g - \Sigma^2\mathbf{j}\big) = 0 . \quad (31)$$

It guarantees the existence of a scalar ω such that

$$\mathrm{d}g = \mathbf{j} + \Sigma^{-2}\operatorname{rot}\omega . \quad (32)$$

However, we have to enforce the integrability conditions

$$\operatorname{rot}\,\mathrm{d}g = 0 , \quad (33)$$

which turns out to be the equation for ω:

$$\operatorname{div}\big(\Sigma^{-2}\mathrm{d}\omega\big) = \operatorname{rot}\mathbf{j} . \quad (34)$$

We thus obtained the decoupled Eqs. (32) and (34) which determine the metric function g.

Substituting $\Sigma^2\mathbf{a} = \operatorname{rot}\tilde{\lambda}$ and (32) to (26), and using identity $b = \operatorname{rot}\big(\Sigma^{-2}\operatorname{rot}\tilde{\lambda}\big)$, we get the decoupled equation for $\tilde{\lambda}$:

$$\frac{1}{2}\triangle\Big(\Sigma^4\operatorname{rot}\big(\Sigma^{-2}\operatorname{rot}\tilde{\lambda}\big)\Big) + 2\varrho_{\mathrm{EM}}\operatorname{rot}\big(\Sigma^{-2}\operatorname{rot}\tilde{\lambda}\big) - \alpha\triangle\tilde{\lambda} = -\triangle\omega . \quad (35)$$

It is a complicated equation of the forth order. It can be simplified to an ordinary differential equation if we assume the additional symmetry properties of the fields, e.g. the rotational symmetry around the axis. The potential $\tilde{\lambda}$ then determines the metric 1-form $\mathbf{a}$ through $\Sigma^2\mathbf{a} = \mathrm{rot}\,\tilde{\lambda}$.

After finding $\mathbf{a}$ one can solve the field equations for $\mathbf{s}$. The potential equations give immediately that

$$\mathbf{s} = \mathrm{d}\varphi\,. \tag{36}$$

Substituting to the condition $\mathrm{div}(\mathbf{s} - \mathbf{a}) = 0$ we get the Poisson equation for φ:

$$\triangle\,\varphi = \mathrm{div}\,\mathbf{a}\,. \tag{37}$$

Finally, the remaining metric function h is determined by the Eq. (29).

4 THE INTERPRETATION OF THE SOLUTIONS

4.1 The geometries of the transversal spacetime

In this section we will investigate the geometry of the transversal metric $\mathbf{q}$ (the wave fronts) (2) and we will determine the constants α, β in the final Eq. (21). Subsequently, we will discuss the various geometries of $\mathbf{q}$ in proper parametrization and we will determine the meaning of the parameter γ.

We impose conditions to the derivatives of Σ (i.e. S) (21), (19) and (18) while using the relation (13) between $\Sigma_{,\rho}$ and S to determine α and β.

First, we impose conditions at the axis $\rho = 0$. We assume that S and $\Sigma_{,\rho}$ vanish at the axis $\rho = 0$, $S = 0$, $\Sigma_{,\rho} = 0$, second, we can always rescale the metric (2) to get $\Sigma|_{\rho=0} = 1$, third, we want no conical singularities there, therefore we assume $\Sigma_{,\rho\rho}|_{\rho=0} = \gamma$, which we can be justified by computation of the ratio of the circumference o divided by 2π times radius in limit $\rho \to 0$,

$$\frac{o}{2\pi r} = \frac{2\pi\,\frac{S}{\Sigma}}{2\pi \int \Sigma\,\mathrm{d}\rho} = \frac{1}{\Sigma}\left(\frac{S}{\Sigma}\right)_{,\rho} = \frac{1}{\gamma}\,\frac{\Sigma_{,\rho\rho}\Sigma - \left(\Sigma_{,\rho}\right)^2}{\Sigma^3} = 1\,. \tag{38}$$

Applying the conditions from last paragraph, we obtain

$$-\frac{1}{3}\Lambda + \alpha + \beta - \varrho_{\mathrm{EM}} = 0\,, \quad -\frac{2}{3}\Lambda + \alpha + \frac{1}{2}\beta = \gamma\,. \tag{39}$$

We can then determine the constants α and β explicitly in terms of the cosmological constant Λ, the density of electromagnetic field ϱ_{EM} and the parameter γ,

$$\alpha = \Lambda - \varrho_{\mathrm{EM}} + 2\gamma\,, \quad \beta = -\frac{2}{3}\Lambda + 2\varrho_{\mathrm{EM}} - 2\gamma\,. \tag{40}$$

We can conveniently rewrite (13),

$$\left(\gamma S\right)^2 = \left(\Sigma_{,\rho}\right)^2 = \left[-\frac{1}{3}\frac{\Lambda}{\gamma^2}\left(\Sigma^2 - 2\right)\Sigma - \frac{\varrho_{\mathrm{EM}}}{\gamma^2}(\Sigma - 1) + \frac{2}{\gamma}\Sigma\right](\Sigma - 1)\,. \tag{41}$$

Now we know explicitly the constants in the derivative of Σ and we can investigate the interpretation of the generalized Melvin spacetime. It is convenient to introduce new coordinate x as

$$\Sigma = 1 + \gamma x \,, \tag{42}$$

then we can write that

$$S = x_{,\rho} \,, \quad \Sigma_{,\rho} = \gamma x_{,\rho} \,. \tag{43}$$

The transversal metric $\mathbf{q}$ (2) then can be rewritten as

$$\mathbf{q} = \left(\frac{\Sigma}{S}\right)^2 \mathrm{d}x^2 + \left(\frac{S}{\Sigma}\right)^2 \mathrm{d}\phi^2 = \frac{1}{G}\,\mathrm{d}x^2 + G\mathrm{d}\phi^2 \,, \tag{44}$$

where we can express the new function G as

$$G = \left(\frac{S}{\Sigma}\right)^2 = -\frac{1}{3}\frac{\Lambda}{\gamma^2}\Sigma^2 + \frac{\alpha}{\gamma^2} + \frac{\beta}{\gamma^2}\frac{1}{\Sigma} - \frac{\varrho_{\mathrm{EM}}}{\gamma^2}\frac{1}{\Sigma^2} \,, \tag{45}$$

and

$$S^2 = \mp\ell^2\gamma^2x^4 \mp \ell^2\gamma x^3 + \big(\mp 3\ell^2 - \varrho_{\mathrm{EM}} + 2\gamma\big)x^2 + 2x, \tag{46}$$

where we denoted $\mp\ell^2 = \Lambda/3$ where $\pm = \operatorname{sign}\Lambda$.

Before we will discuss the possible geometries given by the transversal metric $\mathbf{q}$ (2) and interpret them accordingly we introduce important characteristics for the generalized Melvin spacetime.

The radial radius is then defined as

$$r = \int_0^x \frac{1}{\sqrt{G}}\,\mathrm{d}x \,, \tag{47}$$

the circumference radius is simply given by the function G, $R = \sqrt{G}$. Interestingly, the ratio of the radia is then determined by the derivative of G,

$$\frac{\mathrm{d}R}{\mathrm{d}r} = \sqrt{G}\,\frac{\mathrm{d}\sqrt{G}}{\mathrm{d}x} = \frac{1}{2}G_{,x} \,. \tag{48}$$

The scalar curvature of $\mathbf{q}$ can be also written as

$$\mathcal{R} = -G_{,xx} = -\frac{2}{\Sigma^4}\left[3\Sigma_{,\rho} + \frac{2}{3}\Lambda\Sigma^4 - 3\alpha\Sigma^2 - 2\beta\Sigma\right] . \tag{49}$$

The geometries of the transversal spacetime $\mathbf{q}$ can be illustrated by investigating the function G and its roots when we consider different values of Λ, ϱ_{EM} and of the parameter γ.

First, we consider positive cosmological constant $\Lambda > 0$ for any ϱ_{EM} and γ we obtain *closed space* where $\rho \in (0, \rho_*)$ and ρ_* represents the first positive root of G where in fact the spacetime closes itself. The other characteristics are: the radial radius tends to a finite

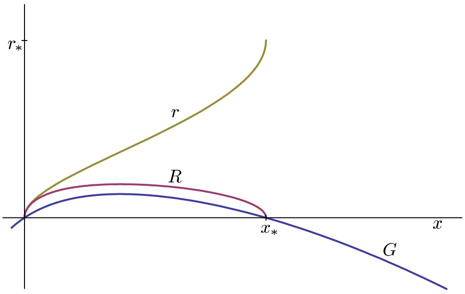

Figure 1. The case when $\Lambda > 0$ which represents closed spacetime. The function G is visualized for any value of ϱ_{EM} and γ. The coordinate ρ ranges $\rho \in (0, \rho_*)$ where the ρ_* is the first root of G where the spacetime closes.

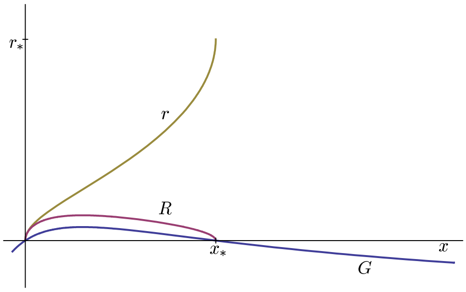

Figure 2. The case when $\Lambda = 0$ and $\varrho_{\mathrm{EM}} > 2\gamma$ represents the closed spacetime. The function G is visualized for $\varrho_{\mathrm{EM}} > 2\gamma$ and the coordinate ρ ranges $\rho \in (0, \rho_*)$ where the ρ_* is the root of G where the spacetime closes.

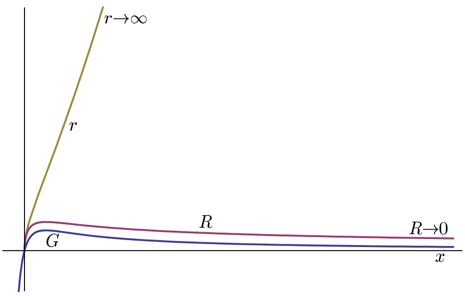

Figure 3. The case when $\Lambda = 0$ and $\varrho_{\mathrm{EM}} = 2\gamma$ then represents the closed spacetime with an infinite peak. The function G is visualized for $\varrho_{\mathrm{EM}} = 2\gamma$ and the coordinate ρ ranges $\rho \to \infty$.

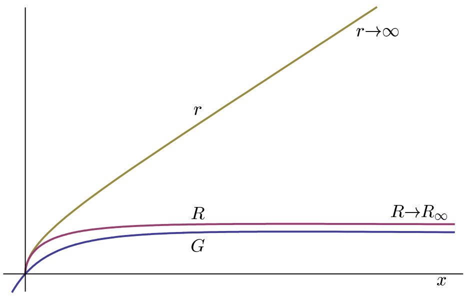

Figure 4. The case when $\Lambda = 0$ and $\varrho_{\mathrm{EM}} < 2\gamma$ then represents the open spacetime. The function G is visualized for $\varrho_{\mathrm{EM}} < 2\gamma$ and the coordinate ρ ranges $\rho \to \infty$.

value $r \to r_*$ at the ρ_* and the circumference radius vanishes $R \to 0$ when $\rho \to \rho_*$. This special case is visualized in the Fig. 1.

For the vanishing cosmological constant $\Lambda = 0$ we obtain three possible spacetimes according to the values of ϱ_{EM} and γ.

When $\varrho_{\mathrm{EM}} > 2\gamma$ then we get *closed space* where the range of the coordinate ρ goes again as $\rho \in (0, \rho_*)$ and ρ_* is then the root of G and it is the closing point of the universe. The radia are then $r \to r_*$ and $R \to 0$ when $\rho \to \rho_*$, see the Fig. 2.

When $\varrho_{\mathrm{EM}} = 2\gamma$ then we obtain *closed space with and infinite peak* for $\rho \to \infty$. Therefore, when $\rho \to \infty$ the radial radius tends to infinity $r \to \infty$ and the circumference radius goes to zero $R \to 0$, see the Fig. 3. This case represents the pure Melvin spacetime Bonnor (1954); Melvin (1965) which we discussed in Kadlecová and Krtouš (2010).

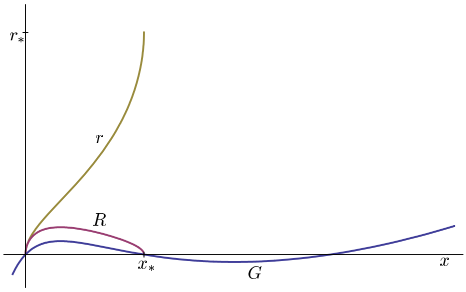

Figure 5. The case when $\Lambda < 0$ and $\gamma < \gamma_{cr}$ represents the closed spacetime. The coordinate ρ ranges $\rho \in (0, \rho_*)$ where the ρ_* is the root of G where the spacetime closes.

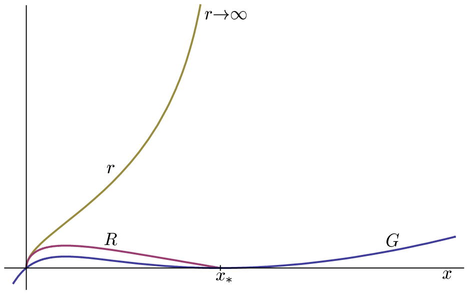

Figure 6. The case when $\Lambda < 0$ and $\gamma = \gamma_{cr}$ represents the asymptotically closed spacetime. The coordinate ρ ranges $\rho \in (0, \rho_*)$ where the ρ_* is the root of G. The radial distance tends to infinity and the circumference shrinks to zero.

Table 1. Possible geometries of the transversal spacetime **q**. Here Λ is a cosmological constant, ϱ_{EM} is energy density of the electromagnetic field and γ is the parameter of 'Melviniztion' of the spacetime. Critical value $\gamma_{cr}(\Lambda, \varrho_{\mathrm{EM}})$ is determined by the condition that the function G has degenerated root at ρ_*.

Λ	$\varrho_{\mathrm{EM}}, \gamma$	transversal spacetime	ρ	$r\vert_{\rho\to\rho_*}$	$R\vert_{\rho\to\rho_*}$
$\Lambda > 0$	any	closed space	$(0, \rho_*)$	r_*	0
	$\gamma < \varrho_{\mathrm{EM}}/2$	closed space	$(0, \rho_*)$	r_*	0
$\Lambda = 0$	$\gamma = \varrho_{\mathrm{EM}}/2$	Melvin universe	$\mathbb{R}^+$	∞	0
	$\gamma > \varrho_{\mathrm{EM}}/2$	open space	$\mathbb{R}^+$	∞	R_∞
	$\gamma < \gamma_{cr}$	closed space	$(0, \rho_*)$	r_*	0
$\Lambda < 0$	$\gamma = \gamma_{cr}$	closed with ∞ peak	$(0, \rho_*)$	∞	0
	$\gamma > \gamma_{cr}$	open space	$\mathbb{R}^+$	∞	∞

When $\varrho_{\mathrm{EM}} < 2\gamma$ then we obtain *an open space* for $\rho \in (0, \infty)$. When $\rho \to \infty$, the radial radius tends to infinity $r \to \infty$; however, the circumference radius goes to a finite value, $R \to R_\infty$, see the Fig. 4.

When we consider the negative cosmological constant $\Lambda < 0$ we obtain three possible spacetimes according to the values of γ. For γ smaller than certain critical value γ_{cr} (which depends on Λ and ϱ_{EM}), we get *closed space* where the range of the coordinate ρ goes again as $\rho \in (0, \rho_*)$ and ρ_* is then the root of G and the closing point of the universe. The radia are then $r \to r_*$ and $R \to 0$ when $\rho \to \rho_*$, see the Fig. 5.

When $\gamma = \gamma_{cr}$, we obtain *closed space with and infinite peak* where the range of the coordinate ρ goes as $\rho \in (0, \rho_*)$ and ρ_* is the root of G. The radia are then $r \to \infty$ and $R \to 0$ when $\rho \to \rho_*$, see the Fig. 6.

When $\gamma > \gamma_{cr}$, we obtain *open space* for $\rho \in (0, \infty)$. For $\rho \to \infty$, $r \to \infty$, and $R \to R_\infty$, see the Fig. 7.

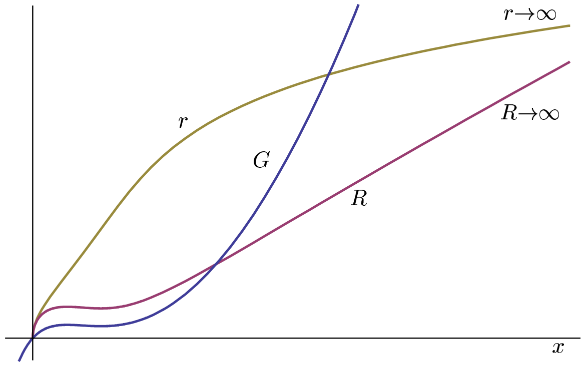

Figure 7. The case when $\Lambda < 0$ and $\gamma > \gamma_{cr}$ represents the open spacetime. The coordinate ρ takes positive real values. For $\rho \to \infty$, $r \to \infty$, and $R \to R_\infty$, see the Fig. 7.

We have summarized our resulting geometries arising from the generalized Melvin universe in a Table 1.

To conclude this section, we have investigated the transversal spacetime of the generalized Melvin universe. We have identified the constants α and β, interpreted them in terms of the cosmological constant Λ, $\varrho_{\rm EM}$ and γ. After suitable parametrization of the transversal spacetime we have discussed all possible cases of universes which are contained in the generalized Melvin universe. The Melvin universe occurs as a special case. We have visualized these cases in figures and summarized them in the Table 1.

The parameter γ changes the character of the influence of the electromagnetic field on the geometry. With larger γ the influence is stronger and for $\Lambda \leq 0$ it can even change the global structure of the spacetime, what exactly happens for the critical value γ_{cr} (for $\Lambda = 0$ $\gamma_{cr} = \varrho_{\rm EM}/2$).

4.2 The backgrounds for our solutions

The background spacetimes are defined as a limit when $h = g = 0$ and $\mathbf{a} = 0$, then the metric (1) reduces to

$$\mathbf{g} = \mathbf{q} - \Sigma^2\, \mathrm{d}u \vee \mathrm{d}v + \alpha v^2 \Sigma^2 \mathrm{d}u^2\,. \tag{50}$$

The metric (50) admits one killing vector ∂_ϕ which corresponds to cylindrical symmetry.

Using the adapted null tetrad $\mathbf{k} = \partial_v$, $\mathbf{l} = \Sigma^{-2}(\partial_u + \alpha v^2 \partial_v/2)$, $\mathbf{m} = (\Sigma^{-1}\partial_\rho - i\,\Sigma S^{-1}\partial_\phi)/\sqrt{2}$, the only non-vanishing components of Weyl and Ricci tensors are,

$$\Psi_2 = \frac{1}{2\Sigma^4}\,(\beta\Sigma - 2\varrho_{\rm EM})\ , \qquad \Phi_{11} = \frac{1}{2\Sigma^4}\varrho_{\rm EM}\,. \tag{51}$$

This demonstrates that the generalized Melvin universe is a non-vacuum solution of type D, except the points where $\Psi_2 = 0$.

Table 2. Some of possible background spacetimes in the case $\gamma = 0$ which represents the direct product of two 2-spaces of constant curvature. The parameter $\Lambda_+ = \Lambda + \varrho_{\rm EM}$ gives the geometry of the wave front and $\Lambda_- = \Lambda - \varrho_{\rm EM}$ determines the conformal structure of the background.

Λ_+	Λ_-	geometry	background	Λ	$\varrho_{\rm EM}$
0	0	$E^2 \times M_2$	Minkowski	$= 0$	$= 0$
Λ	Λ	$S^2 \times dS_2$	Nariai	> 0	$= 0$
Λ	Λ	$H^2 \times AdS_2$	anti-Nariai	< 0	$= 0$
$\varrho_{\rm EM}$	$-\varrho_{\rm EM}$	$S^2 \times AdS_2$	Bertotti–Robinson	$= 0$	> 0
2Λ	0	$S^2 \times M_2$	Plebański–Hacyan	> 0	$= \Lambda$
0	2Λ	$E^2 \times AdS_2$	Plebański–Hacyan	< 0	$= \|\Lambda\|$

The background metric (50) contains several sub-solutions. For $\Lambda = 0$ and $\varrho_{\rm EM} = 2\gamma$ we obtain the Melvin universe which serves as a background in Kadlecová and Krtouš (2010) and the the only non-vanishing Weyl and curvature scalars are

$$\Phi_2 = -\frac{\varrho_{\rm EM}}{2\Sigma^4}(2-\Sigma) = \frac{1}{2}\frac{\varrho_{\rm EM}}{\Sigma^4}\left(-1+\frac{1}{4}\varrho_{\rm EM}\rho^2\right), \quad \Psi_{11} = \frac{1}{2\Sigma^4}\varrho_{\rm EM}\,, \tag{52}$$

where we have used the $\Sigma = 1+\varrho_{\rm EM}\rho^2/4$ which specifies the Melvin spacetime. The scalar curvature of the transversal spacetime $\mathbf{q}$ (49) then becomes

$$\mathcal{R} = 0\,, \tag{53}$$

which agrees with Kadlecová and Krtouš (2010).

For $\Sigma = 1$ we get the direct product background spacetimes, the metric (50) reduces to

$$\mathbf{g} = \mathbf{q} - \mathrm{d}u \vee \mathrm{d}v + \alpha v^2\,\mathrm{d}u^2\,, \tag{54}$$

the only non-vanishing Weyl and curvature scalars then are

$$\Psi_2 = \tfrac{1}{2}\left(\beta - 2\varrho_{\rm EM}\right) = -\tfrac{\Lambda}{3}\,, \quad \Phi_{11} = \tfrac{1}{2}\varrho_{\rm EM}\,. \tag{55}$$

The scalar curvature of the transversal spacetime $\mathbf{q}$ (49) then becomes

$$\mathcal{R} = 2(\Lambda + \varrho_{\rm EM})\,, \tag{56}$$

which agrees with Kadlecová et al. (2009).

To summarize the background metric (50) generalizes the metric for the pure Melvin universe and the direct product spacetimes into one background metric and combines their properties.

5 THE SCALAR POLYNOMIAL INVARIANTS

The scalar invariants are important characteristics of gyraton spacetimes. The gyratons in the Minkowski spacetime Frolov et al. (2005) have vanishing invariants (VSI) Pravda et al. (2002), the gyratons in the AdS Frolov and Zelnikov (2005) and direct product spacetimes

Kadlecová et al. (2009) have all invariants constant (CSI) Coley et al. (2006). The invariants are independent of all metric functions a_i which characterize the gyraton, and have the same values as the corresponding invariants of the background spacetime. We have shown that similar property is valid also for the gyraton on Melvin spacetime Kadlecová and Krtouš (2010), but the invariants are functions of the coordinate ρ and depend on the constant density ϱ_{EM}.

In these cases, the invariants are independent of all metric functions which characterize the gyraton, and have the same values as the corresponding invariants of the background spacetime. We observed that similar property is valid also for the gyraton on Melvin spacetime and it is valid also for its generalization with Λ, however, in this case the invariants are generally *non-constant*, namely, they depend on the coordinate ρ. This property is a consequence of general theorem holding for the relevant subclass of Kundt solution, see Theorem II.7 in Coley et al. (2010). For more details, see Kadlecová (2013).

6 CONCLUSION

Our work generalizes the studies of the gyraton on the Melvin universe Kadlecová and Krtouš (2010). Namely we have generalized the transversal background metric for the pure Melvin universe where instead of the coordinate ρ we have assumed general function S dependent only on the coordinate ρ. This change enabled us to find new solutions with possible non-zero cosmological constant. This is not allowed for the pure Melvin background spacetime. We were able to derive relation between metric functions Σ and S from the source free Einstein–Maxwell equations. The derivative of the function $\Sigma_{,\rho}$ is then polynomial in the function Σ itself and contains four parameters. We have showed that these parameters can be expressed using constants Λ, ϱ_{EM} and γ.

The Einstein–Maxwell equations reduce again to the set of linear equations on the 2-dimensional transverse spacetime which has non-trivial geometry given by the generalized Melvin spacetime (2). Fortunately, these equations do decouple and they can be solved least in principle for any distribution of the matter sources.

In detail, we have studied the transversal geometries of generalized Melvin spacetime (2). We have discussed the various possible values of constants Λ, ϱ_{EM} and γ. It occurs that for $\Lambda > 0$ the transversal geometry represents only one type of space, the case $\Lambda = 0$ includes three different spaces, one of them corresponds to the Melvin spacetime as a special case. The case $\Lambda < 0$ also describes three types of spaces. We have visualized them in several figures in Section 4 and summarized them in the Table 1. Thanks to this discussion we were able to interpret the parameter γ as the parameter which makes the electromagnetic field of the direct product spacetimes stronger.

We have investigated the polynomial scalar invariants. In this generalized case, the invariants are again not constant and they are functions of the metric function Σ and the full gyratonic metric has the same invariants as the background metric.

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