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String loop chaotic scattering in the field of Schwarzschild black hole

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ABSTRACT
Relativistic current-carrying string loop moving axisymmetrically along the axis of a Schwarzschild black hole is investigated as model of relativistic jet formation. Acceleration of the string loop along its axis of symmetry shows regular and also irregular dependence on initial conditions. We will apply the theory of chaotic scattering on this problem.

Keywords: string loop – relativistic jets – chaotic scattering – Schwarzschild – black holes

1 INTRODUCTION

Current-carrying string loop model is relativistic string with circular shape threaded on to black hole axis. Tension of such string loops prevents their expansion beyond some radius, while their worldsheet current introduces an angular momentum barrier preventing them from collapsing into the black hole. The string loop oscillates in the x-z plane propagating simultaneously in the y-direction. Such model could in a simplified way represent plasma that exhibits associated string-like behaviour via dynamics of the magnetic field lines in the plasma (Semenov et al., 2004) or due to thin isolated flux tubes of magnetized plasma that could be described by an one-dimensional string (Spruit, 1981; Semenov and Bernikov, 1991).

From the astrophysical point of view, one of the most relevant applications of the axisymmetric string loop motion is the possibility of strong acceleration of the linear translational string loop motion due to the transmutation process in the strong gravity of extremely compact objects that could well mimic acceleration of relativistic jets in Active Galactic Nuclei (AGN) and microquasars (Jacobson and Sotiriou, 2009). Due to chaotic nature of string loop equation of motion (Frolov and Larsen, 1999), the resulting acceleration in the terms of the translational velocity or gamma factor shows strong dependence on the initial conditions (Stuchlík and Kološ, 2009). In this article we would like to address this problem from the point of view of chaotic scattering theory as it is presented in Chapter 5 of Ott (1993) and Chapter 8 in Tél and M. (2006).

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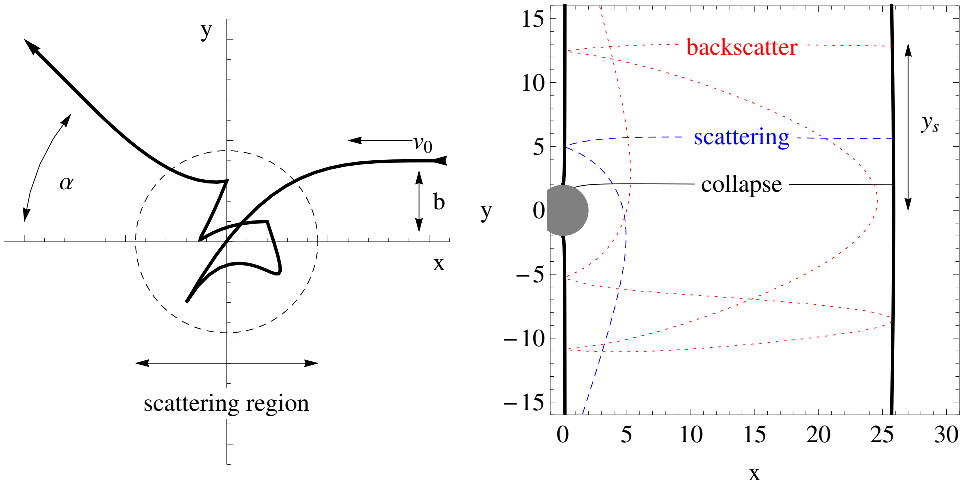

Figure 1. Schematic picture of standard chaotic scattering of particle moving towards scattering region of effective potential (*left*) and chaotic scattering of string loop on Schwarzschild black hole (*right*). On the right, in the case of string loop, we assumed axial symmetry which allows to investigate only one point of the loop; one point path can represent whole string loop movement. Trajectory of the loop is then represented by the black curve on the picture, given in 2D x-y plot. If the string loop is in equatorial plane $y = 0$, its overall loop circle will be seen in x-z plane.

Let us we have a particle with impact parameter b, entering some part of effective potential (scattering region), and let's the particle can escape the scattering region with some scattering angle α, see Fig. 1 (left). We can define scattering angle (scattering function) $\alpha(b)$ as a function depending on impact parameter b. Chaotic scattering theory is dealing with the properties of scattering function $\alpha(b)$, especially when $\alpha(b)$ shows some "strange" (chaotic behaviour). In our system of string loop winding around black hole, we will shoot string loops from some position y_s giving initial distance from the equatorial plane (impact parameter), towards to the black hole (effective potential) and we will measure final gamma factor γ (scattering angle), see Fig. 1 (right). Properties of the scattering function $\gamma(y_s)$ for string loop dynamics in the vicinity of Schwarzschild black hole are examined in this report.

2 CURRENT-CARRYING STRING LOOP

We study a string loop motion in the field of a black hole described by the Schwarzschild metric

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + A^{-1}(r)\,\mathrm{d}r^2 + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big), \qquad A(r) = 1 - \frac{2M}{r}\,. \tag{1}$$

We use the geometric units with $c = G = 1$ and the Schwarzschild coordinates. In order to properly describe the string loop motion, it is useful to use the Cartesian coordinates

$$x = r\sin(\theta), \qquad y = r\cos(\theta)\,. \tag{2}$$

The string loop is threaded on to an axis of the black hole chosen to be the y-axis. Due to the assumed axisymmetry of the string motion one point path can represent whole movement of the string. Trajectory of the string can be represented by a curve in the 2D x-y plane, see Fig. 1 (right). The string loop can oscillate, changing its radius in the x-z plane, while propagating in the y direction.

The string loop motion is governed by barriers given by the string tension and the worldsheet current determining the angular momentum – these barriers are modified by the gravitational field. Dynamics of the string is described by the action

$$S = \int \mathrm{d}^2\sigma \sqrt{-h}\big(\mu + h^{ab}\varphi_{,a}\varphi_{,b}\big)\,, \tag{3}$$

where $\varphi_{,a} = j_a$ determines current of the string and $\mu > 0$ reflects the string tension. Axisymmetry of the string loop means that the scalar field $\varphi = j_\sigma \sigma + j_\tau \tau$, where j_σ and j_τ are constant components of the current.

The worldsheet stress-energy tensor density $\tilde{\Sigma}^{ab}$ can be expressed in the form (Jacobson and Sotiriou, 2009)

$$\tilde{\Sigma}^{\tau\tau} = \frac{J^2}{g_{\phi\phi}} + \mu\,, \quad \tilde{\Sigma}^{\sigma\sigma} = \frac{J^2}{g_{\phi\phi}} - \mu\,, \quad \tilde{\Sigma}^{\sigma\tau} = \frac{-2 j_\tau j_\sigma}{g_{\phi\phi}}\,, \quad J^2 \equiv j_\sigma^2 + j_\tau^2\,. \tag{4}$$

As demonstrated in (Larsen, 1993; Carter and Steer, 2004), the string loop motion in spherically symmetric spacetime can be described by the Hamiltonian

$$H = \frac{1}{2} g^{rr}\, P_r^2 + \frac{1}{2} g^{\theta\theta} P_\theta^2 + \frac{1}{2} g_{\phi\phi} \left(\Sigma^{\tau\tau}\right)^2 + \frac{1}{2} g^{tt} E^2\,. \tag{5}$$

The equations of motion are given by the Hamilton equations

$$\frac{\mathrm{d}X^\mu}{\mathrm{d}\zeta} = \frac{\partial H}{\partial P_\mu}\,, \quad \frac{\mathrm{d}P_\mu}{\mathrm{d}\zeta} = -\frac{\partial H}{\partial X^\mu}\,. \tag{6}$$

Due to symmetries of metrics (1) we have conserved quantities string loop energy E and string loop angular momentum L, given by

$$-E = P_t = g_{tt} \tilde{\Sigma}^{\tau\tau} X^t_{|\tau}\,, \quad L = P_\phi = g_{\phi\phi} \tilde{\Sigma}^{\sigma\tau} = -2 j_\tau j_\sigma\,. \tag{7}$$

Hamiltonian is constant of the motion $H = 0$. The loci where the string loop has zero velocity ($\dot{r} = 0, \dot{\theta} = 0$) form boundary of the string motion

$$E = E_{\mathrm{b}}(r,\theta) = \sqrt{-g_{tt} g_{\phi\phi}}\; \tilde{\Sigma}^{\tau\tau}\,. \tag{8}$$

There are four different types of the behaviour of the energy boundary function for the string loop dynamics in the Schwarzschild BH spacetime represented by the characteristic $E =$ const. sections of the $E_{\mathrm{b}}(r,\theta)$ function in dependence on parameter J (Jacobson and Sotiriou, 2009). We can distinguish them according to two properties: possibility of the string loop to escape to infinity in the y-direction, and possibility to collapse to the black hole. A detailed discussion can be found in Kološ and Stuchlík (2010).

The first case corresponds to no inner and outer boundary – the string loop can be captured by the black hole or escape to infinity. The second case corresponds to the situation with an outer boundary – the string loop must be captured by the black hole. The third case corresponds to the situation when both inner and outer boundary exist – the string loop is trapped in some region forming a potential "lake" around the black hole. The fourth case corresponds to an inner boundary – the string loop cannot fall into the black hole but it must escape to infinity, see Fig. 2 in Stuchlík and Kološ (2009). For our following discussion only the first and fourth case, corresponding to the possibility of the string loop to escape to infinity in the y-direction, will be relevant.

3 STRING LOOP ASYMPTOTICAL EJECTION SPEED

Since the Schwarzschild spacetime is asymptotically flat, we will discuss the string loop motion in the flat spacetime that enables clear definition of the acceleration process. The energy of the string loop (8) in the flat spacetime, expressed in the Cartesian coordinates, reads

$$E^2 = \dot{y}^2 + \dot{x}^2 + \left(\frac{J^2}{x} + x\right)^2 = E_{\mathrm{y}}^2 + E_{\mathrm{x}}^2 \,, \tag{9}$$

where dot denotes derivative with respect to the affine parameter ζ. The energy related to the motion in the x- and y-directions are given by the relations

$$E_{\mathrm{y}}^2 = \dot{y}^2, \quad E_{\mathrm{x}}^2 = \dot{x}^2 + \left(\frac{J^2}{x} + x\right)^2 = (x_{\mathrm{i}} + x_{\mathrm{o}})^2 = E_0^2 \tag{10}$$

where x_i (x_o) represent the inner (outer) limit of the oscillatory motion. The energy E_0 representing the internal energy of the string loop is minimal when the inner and the outer radii coincide, leading to the relation

$$E_{0(\mathrm{min})} = 2J \tag{11}$$

that determines the minimal energy necessary for escaping of the string loop to infinity. Clearly, $E_{\mathrm{x}} = E_0$ and E_{y} are constants of the string loop motion and no transformation between these energy modes is possible in the flat spacetime. However, in strong gravity in vicinity of black holes or naked singularities, the internal kinetic energy of the oscillating string can be transmitted into the kinetic energy of the translational linear motion (or vice versa) due to the chaotic character of the string loop dynamics (Jacobson and Sotiriou, 2009; Stuchlík and Kološ, 2012a).

In order to get a strong acceleration in the Schwarzschild spacetime, the string loop has to pass the region of strong gravity near the black hole horizon (scattering region), where the string transmutation effect $E_{\mathrm{x}} \leftrightarrow E_{\mathrm{y}}$ can occur. All energy of the transitional (E_{y}) energy mode can be transmitted to the oscillatory (E_{x}) energy mode – oscillations of the string loop in the x-direction and the internal energy of the string will increase maximally in such a situation, while the string will stop moving in the y-direction. However, all energy of

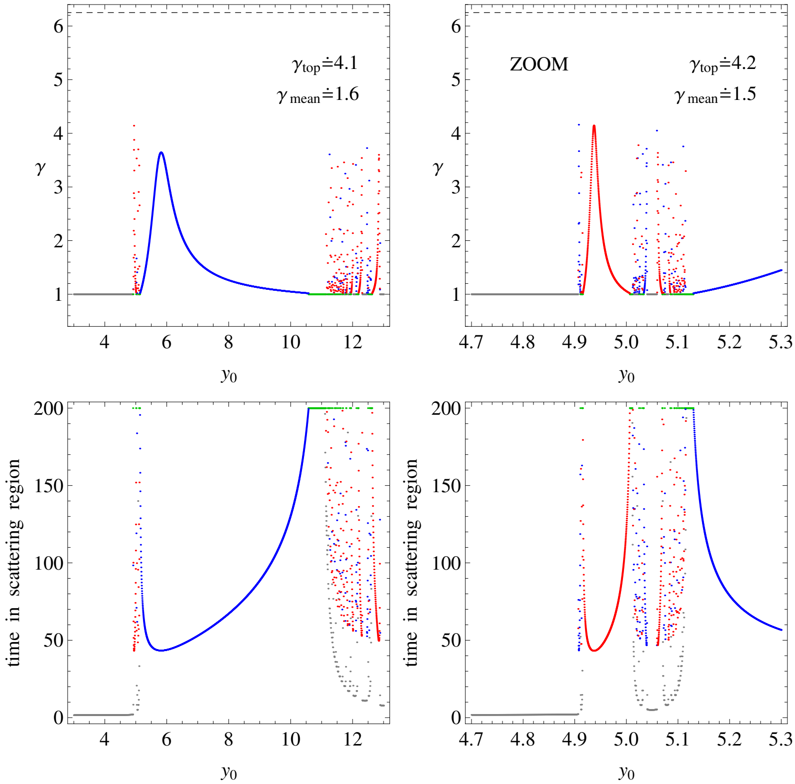

Figure 2. Scattering function $\gamma(y_s)$ (Lorentz factor at infinity) and time spend by the string loop in the region close to the black hole horizon (scattering region) is calculated for energy $E = 25$ and current $J = 2$. All trajectories starting from the rest with different initial position $y_0 \in (3, 13)$ while x_0 is calculated from E_b condition (8). Gray points correspond to the string loops collapsed to the black hole, blue to the scattered and red backscattered string loops. Green are trajectories which were not able to reach numerical infinity located at $r = 1000$ in given maximal integration time $\zeta = 200$. Examples of individual trajectories trajectories can be found in Fig. 1. Maximal acceleration for this case (13) gives us the limiting gamma factor $\gamma_{max} = 6.25$ (*dashed line*). We show the topical gamma factor that is numerically found in the sample, γ_{top}, and also the mean value γ_{mean} from the sample. Figure on the left is only zoom in to the figure on the right for values $y_0 \in (4.7, 5.3)$ (first chaotic band).

the E_x mode cannot be transmitted into the E_y energy mode – there remains inconvertible internal energy of the string, $E_{0(min)} = 2J$, being the minimal potential energy hidden in the E_x energy mode.

The final Lorentz factor of the transitional motion of an accelerated string loop as observed in the asymptotically flat region of the Schwarzschild spacetime is, due to (10), determined by the relation (Jacobson and Sotiriou, 2009; Stuchlík and Kološ, 2012a)

$$\gamma = \frac{E}{E_0} = \frac{E}{x_i + x_o}, \qquad (12)$$

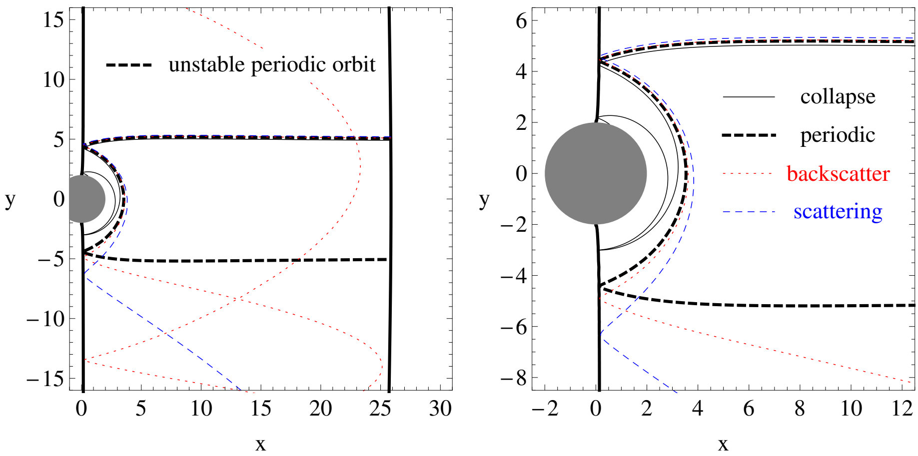

Figure 3. String loop unstable periodic orbit (UPO) compared to orbits obtained by slight change in UPO initial conditions. Presented UPO, with initial starting position $y_{\mathrm{s}} \doteq 5.05905$, is responsible for first chaotic band on left Fig. 2.

where E is the total energy of the string loop moving with the internal energy E_0 in the y-direction with the velocity corresponding to the Lorentz factor γ.

To see how the acceleration of the string loop in the field of Schwarzschild black hole works, we will start to "shoot" string loops from position $x_{\mathrm{s}} \sim 25$, $y_{\mathrm{s}} \in (3, 13)$ with energy $E = 25$ and current $J = 2$, see Fig. 1 (right). Maintaining the string loop energy $E = 25$ constant for all trajectories of $y_{\mathrm{s}} \in (3, 13)$ sample, we must calculate the starting coordinate x_{s} from energy condition (8). For every starting position y_{s} (impact parameter) we will measure final gamma factor, γ given by (12) and hence obtaining the gamma factor $\gamma(y_{\mathrm{s}})$ (scattering function) as function of starting position.

As can be seen from Fig. 2., the scattering function $\gamma(y_{\mathrm{s}})$ have some regular scattering regions, example is the region $y_{\mathrm{s}} \in (6, 10)$, where the final γ factor is changing continuously with initial starting position y_{s}. Such behaviour is expected by common sense, because it is observed in many normal (non chaotic) scatterings. But for chaotic scattering there are also chaotic regions (chaotic bands), example is the region $y_{\mathrm{s}} \in (5.0, 5.1)$, where it is not possible to predict final γ factor output from neighbouring initial starting points $y_{\mathrm{s}} \pm \delta$ – the scattering function $\gamma(y_{\mathrm{s}})$ is not continuous.

To find the origin of chaotic bands in our system, we can compare the scattering function $\gamma(y_{\mathrm{s}})$ (upper row of pictures in Fig. 2) with the the integration time which the string loop is spending in region close to the black hole horizon (scattering region) before escaping to the infinity (lower row of pictures in Fig. 2). Now it is obvious, that trajectories from chaotic bands are spending large amount of time in region close to the black hole; many time crossing the equatorial plane in attempt to decide in which direction to go. The origin of such string loop motion lies in the existence of the unstable periodic orbits (UPOs) in the system, (Ott, 1993; Tél and M., 2006).

String loop at an unstable periodic orbit (UPO) will forever periodically oscillate close to black hole horizon and never leave it, even if there is possibility for escape to infinity from the energetic point of view (energy boundary function E_{b} is open to infinity in y

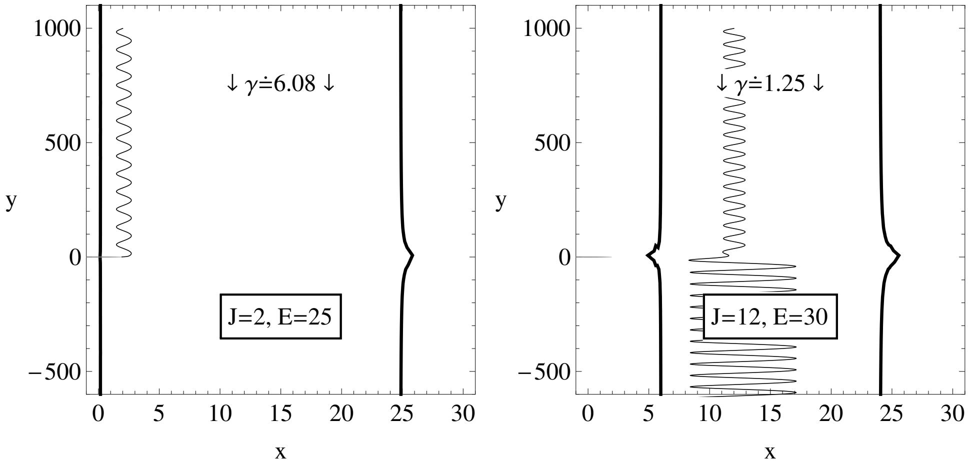

Figure 4. Particle is coming from infinity with almost maximal speed $\gamma \sim \gamma_{\rm max}$. On the left we have the case of motion for small values of current parameter $J = 2$ (1st type of energy boundary) when the sting loop can collapse to the black hole, while on the right we have $J = 12$ (4th type of energy boundary) when the string loop collapse is prohibited.

direction), see Fig. 3. But if the initial conditions for UPO are only sightly changed, the string loop trajectory are completely different, see Fig. 3. We can not predict final output from neighbouring initial conditions.

4 MAXIMAL EJECTION SPEED

During the acceleration, the energy of the oscillatory mode $E_{\rm x}$ is transmitted into translational energy E_y, but there always remains inconvertible internal energy of the string, $E_{0({\rm min})} = 2J$ (11), in the $E_{\rm x}$ mode. This gives limit on string loop maximal acceleration, there exist the maximal Lorentz factor for string loop ejection speed as shown in (Stuchlík and Kološ, 2012a)

$$\gamma_{\rm max} = \frac{E}{2J} \,. \tag{13}$$

From this equation we see that large ratio of the string loop energy E versus its angular momentum given by the current parameter J is needed for ultra-relativistic acceleration. We can use small values of parameter J or large string loop energy E.

We have calculated 3000 trajectories for string loops with energy $E = 25$ and current $J = 2$, with limiting gamma factor $\gamma_{\rm max} = 6.25$ (13), but observed top accelerated string loop has only $\gamma_{\rm top} \doteq 4.2$, see Fig. 2. No trajectory with extreme acceleration $\gamma \sim \gamma_{\rm max}$ was found.

To see for better resolution in Fig. 2, if there can exist an extremely accelerated string loop, hidden somewhere in the chaotic bands, we will examine more closely how such trajectory will looks like. For $\gamma = \gamma_{\rm max}$ the string loop will stop oscillating in the x

direction and moves only along the y axis, with constant radius $x_i = x_o = J$, see (12). Since the string loop motion is time reversible $t \leftrightarrow -t$, instead of escape, we will consider string loop with $\gamma \sim \gamma_{max}$ coming from the infinity towards to the black hole.

Now we can have different situations, depending on the value of the current parameter J, see Fig. 4. If J is quite small, $J \sim 3$ or smaller, the string loop will collapse to the black hole horizon. Loops with $\gamma \sim \gamma_{max}$ have very tiny oscillations in x direction and hence can't "jump over" black hole. Obviously this is the reason why we do not see extremely accelerated string loop $\gamma \sim \gamma_{max}$ in Fig. 2 – such a trajectory had to be started from the black hole. However, such a situation can occur on the naked-singularity spacetimes (Stuchlík and Kološ, 2012b), where the region of strong gravity is not hidden by the event horizon, and $\gamma \sim \gamma_{max}$ can be obtained (Kološ and Stuchlík, 2013). If parameter J is large, typically $J > 10$, extremely accelerated trajectories can not collapse to the black hole (it is prevented by energetic conditions) but they are also too far from gravity well where the string loop transmutation process occur.

5 CONCLUSIONS

The existence of chaotic bands in the scattering function $\gamma(y_s)$ is given by presence of unstable periodic orbit in the system. There exists energetic limit on the maximal string loop acceleration γ_{max}. Large string acceleration along the y-axis can occurs only for large $E/2J$ ratios. It is easy to observe extremely accelerated string loop $\gamma \sim \gamma_{max}$ in the case of naked singularity spacetime, where the horizon is missing.

It should be stressed that rotation of the black hole (naked singularity) is not a relevant ingredient of the acceleration of the string loop motion due to the transmutation effect (Stuchlík and Kološ, 2012a), contrary to the Blandford–Znajek effect (Blandford and Znajek, 1977) usually considered in modelling acceleration of jet-like motion in AGN and microquasars.

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