Transition from regular to chaotic string loop motion

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ABSTRACT
We study transition from regular to chaotic motion in the neighbourhood of stable equilibrium point of a relativistic current-carrying string-loop located around Schwarzschild black hole. We demonstrate successive transfer from the purely regular, periodic motion through quasi-periodic motion to purely chaotic motion of the string loop, with increasing of its energy. We also calculated quasi-periodic fundamental frequencies, which are important for survival of corresponding KAM tori. Using maximal Lyapunov exponent we show how the chaoticity of the string loop motion changes with increase of the string loop energy.

Keywords: chaos and regularity – string loop – Schwarzschild – black holes – Lyapunov exponent

1 INTRODUCTION

Relativistic current-carrying strings moving axisymmetrically along the axis of a Kerr black hole have been studied in (Jacobson and Sotiriou, 2009) where it has been proposed that such a string loop configuration can be used as a model of jet formation and acceleration in the field of black holes in microquasars or active galactic nuclei. Tension of such string loops prevents their expansion beyond some radius, while their worldsheet current introduces an angular momentum barrier preventing them from collapsing into the black hole. It has bee shown that string loop model could in a simplified way represent plasma that exhibits associated string-like behaviour via dynamics of the magnetic field lines in the plasma (Christensson and Hindmarsh, 1999; Semenov et al., 2004) or due to thin isolated flux tubes of magnetized plasma that could be described by an one-dimensional string (Spruit, 1981; Semenov and Bernikov, 1991; Cremaschini and Stuchlík, 2013).

The astrophysical applications of the current carrying string loops have been focused on the problem of acceleration of string loops due to the transmutation process (Jacobson and Sotiriou, 2009), the role of the cosmic repulsion in the string loop motion has been investigated for the Schwarzschild–de Sitter (SdS) spacetime in (Kološ and Stuchlík, 2010a). Since the string loops can be accelerated to ultra-relativistic velocities in the deep gravitational potential well of compact objects (Stuchlík and Kološ, 2009; Kološ and Stuchlík, 2010b; Stuchlík and Kološ, 2012a,b), the string loop transmutation can be well considered as a

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process of formation of ultra-relativistic jets, along with the standard model based on the Blandford–Znajek process (Blandford and Znajek, 1977). Here we concentrate out attention on the inverse situation of small oscillations of string loops in vicinity of stable equilibrium points in the equatorial plane of black holes that was proposed as a possible model of HF QPOs observed in black hole and neutron star binary systems (Stuchlík and Kološ, 2012b).

2 CURRENT-CARRYING STRING LOOP MOTION

We study a string loop motion in the field of a black hole described by the Schwarzschild metric

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + A^{-1}(r)\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \quad A(r) = 1 - \frac{2M}{r}\,. \tag{1}$$

We use the geometric units with $c = G = 1$ and the Schwarzschild coordinates. In order to properly describe the string loop motion, it is useful to use the Cartesian coordinates

$$x = r\sin(\theta), \quad y = r\cos(\theta)\,. \tag{2}$$

The string loop is threaded on to an axis of the black hole chosen to be the y-axis. Due to the assumed axisymmetry of the string motion one point path can represent whole movement of the string. Trajectory of the string can be represented by a curve in the 2D x-y plane. The string loop can oscillate, changing its radius in x-z plane, while propagating in y direction.

The string loop motion is governed by barriers given by the string tension and the worldsheet current determining the angular momentum – these barriers are modified by the gravitational field. Dynamics of the string is described by the action

$$S = \int \mathrm{d}^2\sigma\sqrt{-h}(\mu + h^{ab}\varphi_{,a}\varphi_{,b})\,, \tag{3}$$

where $\varphi_{,a} = j_a$ determines current of the string and $\mu > 0$ reflects the string tension.

The worldsheet stress-energy tensor density $\tilde{\Sigma}^{ab}$ can be expressed in the form (Jacobson and Sotiriou, 2009)

$$\tilde{\Sigma}^{\tau\tau} = \frac{J^2}{g_{\phi\phi}} + \mu\,, \quad \tilde{\Sigma}^{\sigma\sigma} = \frac{J^2}{g_{\phi\phi}} - \mu\,, \quad \tilde{\Sigma}^{\sigma\tau} = \frac{-2j_\tau j_\sigma}{g_{\phi\phi}}\,, \quad J^2 \equiv j_\sigma^2 + j_\tau^2\,. \tag{4}$$

We shall use for simplicity the dimensionless radial coordinate $r/M \to r$, dimensionless time coordinate $t/M \to t$, and we make the rescaling $E_\mathrm{b}/\mu \to E_\mathrm{b}$ and $J/\sqrt{\mu} \to J$.

As demonstrated in (Larsen, 1993), the string loop motion in spherically symmetric spacetimes can be described by the Hamiltonian

$$H = \frac{1}{2}g^{rr}\,P_r^2 + \frac{1}{2}g^{\theta\theta}P_\theta^2 + \frac{1}{2}g_{\phi\phi}\left(\Sigma^{\tau\tau}\right)^2 + \frac{1}{2}g^{tt}E^2\,. \tag{5}$$

The motion of string loops is given by the Hamilton equations in the form

$$\frac{\mathrm{d}X^\mu}{\mathrm{d}\zeta} = \frac{\partial H}{\partial P_\mu}\,, \quad \frac{\mathrm{d}P_\mu}{\mathrm{d}\zeta} = -\frac{\partial H}{\partial X^\mu}\,, \tag{6}$$

where X^μ is 4-position, P^μ is the 4-momentum and ζ is the affine parameter.

Due to symmetries of metric (1), conserved quantities occur for the string loop motion, being the energy E and string the axial angular momentum L, given by

$$-E = P_t = g_{tt}\tilde{\Sigma}^{\tau\tau}X^t_{|\tau}\,, \quad L = P_\phi = g_{\phi\phi}\tilde{\Sigma}^{\sigma\tau} = -2j_\tau j_\sigma\,. \tag{7}$$

The components of the current, j_τ, j_σ, give the angular momentum of the string loop (Stuchlík and Kološ, 2012a).

Hamiltonian is constant of the motion, $H = 0$. The loci where the string loop has zero velocity ($\dot{r} = 0, \dot{\theta} = 0$) form boundary of the string motion

$$E = E_{\mathrm{b}}(r,\theta) = \sqrt{-g_{tt}g_{\phi\phi}}\,\tilde{\Sigma}^{\tau\tau}\,. \tag{8}$$

Function $E_{\mathrm{b}}(r,\theta)$ is playing the role of effective potential, see discussion in (Stuchlík and Kološ, 2012a), its shape is determined by current parameter $J^2 = j_\tau^2 + j_\sigma^2$.

There are four different types of the behaviour of the energy boundary function for the string loop dynamics in the Schwarzschild BH spacetime represented by the characteristic $E =$ const sections of the function $E_{\mathrm{b}}(r,\theta)$ in dependence on parameter J (Jacobson and Sotiriou, 2009). We can distinguish them according to two properties: possibility of the string loop to escape to infinity in the y-direction, and possibility to collapse to the black hole. A detailed discussion can be found in Kološ and Stuchlík (2010a), here we shortly summarize the results.

The first case corresponds to no inner and outer boundary – the string loop can be captured by the black hole or escape to infinity. The second case corresponds to the situation with an outer boundary – the string loop must be captured by the black hole. The third case corresponds to the situation when both inner and outer boundary exist – the string loop is trapped in some region forming a potential "lake" around the black hole. The fourth case corresponds to an inner boundary – the string loop cannot fall into the black hole but it must escape to infinity, see Fig. 2. in Stuchlík and Kološ (2009). For our following discussion only the third case, corresponding to the string loop trapped in toroidal space along black hole, will be relevant.

3 SMALL OSCILLATIONS AROUND MINIMA OF THE "EFFECTIVE POTENTIAL"

It is convenient to examine systems which are constructed from regular part, H_0, plus some small non-linear perturbation, H_{p},

$$H = H_0 + \epsilon H_{\mathrm{p}}\,. \tag{9}$$

As the non-linear parameter ϵ increases, it causes a non-linearity in the system. This "regular+perturbation" separation in not possible in every given Hamiltonian, examples can be given by string loop model (5), or by charged particles moving in combined magnetic and gravitational field, (Kopáček et al., 2010).

However the "regular+perturbation" separation (9) of the Hamiltonian can be done in the neighbourhood of any elliptic point of the Hamiltonian, (Arnold, 1978; Tabor, 1989). The equilibrium points of the Hamiltonian (5) correspond to the local minima at $X^\alpha_0 = (r_0, \theta_0)$

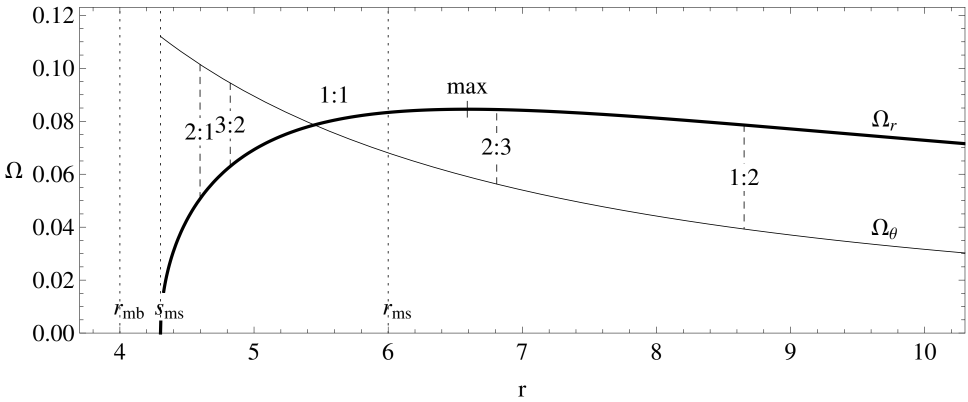


Figure 1. Fundamental frequencies $\Omega_\mathrm{r}(r)$ and $\Omega_\theta(r)$, as function of radial coordinate r, for string loop oscillations in equatorial plane of Schwarzschild BH. Resonant and another important radii, such as marginally stable $r_\mathrm{ms} = 6$, marginally bound $r_\mathrm{mb} = 4$ orbit for particle motion and marginally stable $s_\mathrm{ms} \doteq 4.3$ string loop position, are also given.

of the energy boundary function $E_\mathrm{b}(r,\theta)$, (Arnold, 1978). It is useful to rewrite the Hamiltonian in the form

$$H = H_\mathrm{D} + H_\mathrm{P} = \frac{1}{2} g^{rr} P_r^2 + \frac{1}{2} g^{\theta\theta} P_\theta^2 + H_P(r,\theta) \tag{10}$$

where we split H into the "dynamical" H_D and the "potential" H_P parts. Introducing a small parameter $\epsilon \ll 1$, we can rescale coordinates and momenta by the relations

$$X^\alpha = X^\alpha_0 + \epsilon \hat{X}^\alpha \,, \quad P_\alpha = \epsilon \hat{P}_\alpha \,, \tag{11}$$

applied for the coordinates $\alpha \in \{r,\theta\}$. We can make polynomial expansion of the Hamiltonian into the Taylor series and express it in separated parts according to the power of ϵ

$$H\left(\hat{P}_\alpha, \hat{X}^\alpha\right) = H_0 + \epsilon H_1\left(\hat{X}^\alpha\right) + \epsilon^2 H_2\left(\hat{P}_\alpha, \hat{X}^\alpha\right) + \epsilon^3 H_3\left(\hat{P}_\alpha, \hat{X}^\alpha\right) + \cdots \,, \tag{12}$$

where H_k is a homogeneous part of the Hamiltonian of degree k considered for the momenta $\hat{P}_\alpha$ and coordinates $\hat{X}^\alpha$. Recall that P_α occurs in the quadratic form in (5) and appears in H_k only for $k \geq 2$. If the string loop is located at a local minimum of the $E_\mathrm{b}(x,y)$ function, we have $H_\mathrm{D} = 0$ and hence $H_0 = 0$. The local extrema of the E_b function, given by (6), imply also $H_1(\hat{X}^\alpha) = 0$.

We can divide (12) by the factor ϵ^2 (remember $H = 0$) expressing the Hamiltonian in the vicinity of the local minimum in the "regular" plus "perturbation" form

$$H = H_2\left(\hat{P}_\alpha, \hat{X}^\alpha\right) + \epsilon H_3\left(\hat{P}_\alpha, \hat{X}^\alpha\right) + \ldots \tag{13}$$

If $\epsilon = 0$, we arrive to an integrable Hamiltonian

$$H = H_2\left(\hat{P}_\alpha, \hat{X}^\alpha\right) = \frac{1}{2} \sum_\alpha \left[g^{\alpha\alpha} \left(\hat{P}_\alpha\right)^2 + \tilde{\omega}_\alpha^2 \left(\hat{X}^\alpha\right)^2 \right] \tag{14}$$

representing two uncoupled harmonic oscillators. This "perturbation" approach corresponds to the linearisation of the motion Eqs. (6) in the neighbourhood of local minima of the function $E_{\mathrm{b}}(r, \theta)$.

For the string loop motion represented by coordinates $r = r_0 + \delta r$, $\theta = \theta_0 + \delta\theta$ we obtain the periodic harmonic oscillations determined by the equations

$$\ddot{\delta r} + \omega_{\mathrm{r}}^2 \, \delta r = 0 \,, \quad \ddot{\delta \theta} + \omega_{\theta}^2 \, \delta\theta = 0 \,, \tag{15}$$

where the locally measured frequencies of the oscillatory motion are given by

$$\omega_{\mathrm{r}}^2 = \frac{1}{g_{rr}} \frac{\partial^2 H_{\mathrm{P}}}{\partial r^2} \,, \quad \omega_{\theta}^2 = \frac{1}{g_{\theta\theta}} \frac{\partial^2 H_{\mathrm{P}}}{\partial \theta^2} \,. \tag{16}$$

The locally measured angular frequencies

$$\omega_{(\mathrm{r},\theta)} = \frac{\mathrm{d} f_{(\mathrm{r},\theta)}}{\mathrm{d}\zeta} \tag{17}$$

are connected to the angular frequencies related to distant observers, Ω, by the gravitational redshift transformation

$$\Omega_{(\mathrm{r},\theta)} = \frac{\mathrm{d} f_{(\mathrm{r},\theta)}}{\mathrm{d}t} = \frac{\omega_{(\mathrm{r},\theta)}}{P^t} \,, \tag{18}$$

where $P^t = \mathrm{d}\, t \,/\, \mathrm{d}\, \zeta = -g^{tt} E$. If the angular frequencies $\Omega_{(\mathrm{r},\theta)}$, or frequencies $\nu_{(\mathrm{r},\theta)}$, of the string loop oscillation are expressed in the physical units, their dimensionless form has to be extended by the factor c^3/GM. Then the frequencies of the string loop oscillations measured by the distant observers are given by

$$\nu_{(\mathrm{r},\theta)} = \frac{1}{2\pi} \frac{c^3}{GM} \, \Omega_{(r,\theta)} \,. \tag{19}$$

Notice that this is the same factor as the one occurring in the case of the orbital and epicyclic frequencies of the geodetical motion in the black hole spacetimes, (Török and Stuchlík, 2005). Therefore, the order of magnitude and scaling of the frequencies of the radial and vertical oscillations due to the mass of the central object is the same for both current-carrying string loops and test particles.

In the Schwarzschild spacetime the harmonic oscillations have frequencies (16) relative to distant observers given by expressions relatively very simple for both string loops and test particles. Therefore, we can give the frequencies in dimensional form, as an example. In the case of string loops they read

$$\Omega_{\mathrm{r}}^2(r) = \frac{3M^2 - 5Mr + r^2}{r^4} \,, \quad \Omega_{\theta}^2(r) = \frac{M}{r^3} \,, \tag{20}$$

while for the epicyclic motion of test particles there is

$$\Omega_{\mathrm{r(geo)}}^2(r) = \frac{M(r - 6M)}{r^4} \,, \quad \Omega_{\theta(\mathrm{geo})}^2(r) = \frac{M}{r^3} \,. \tag{21}$$

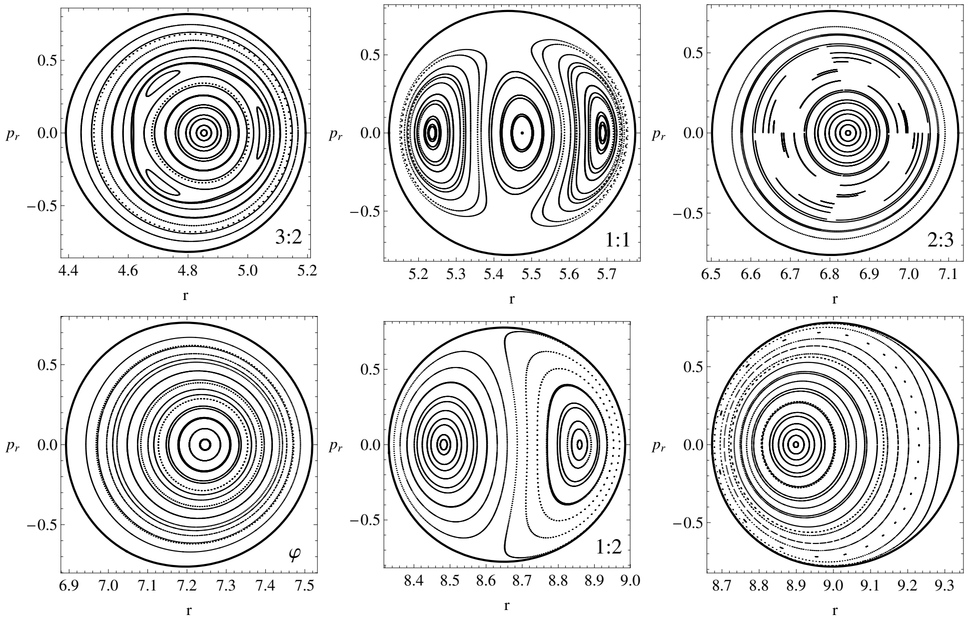


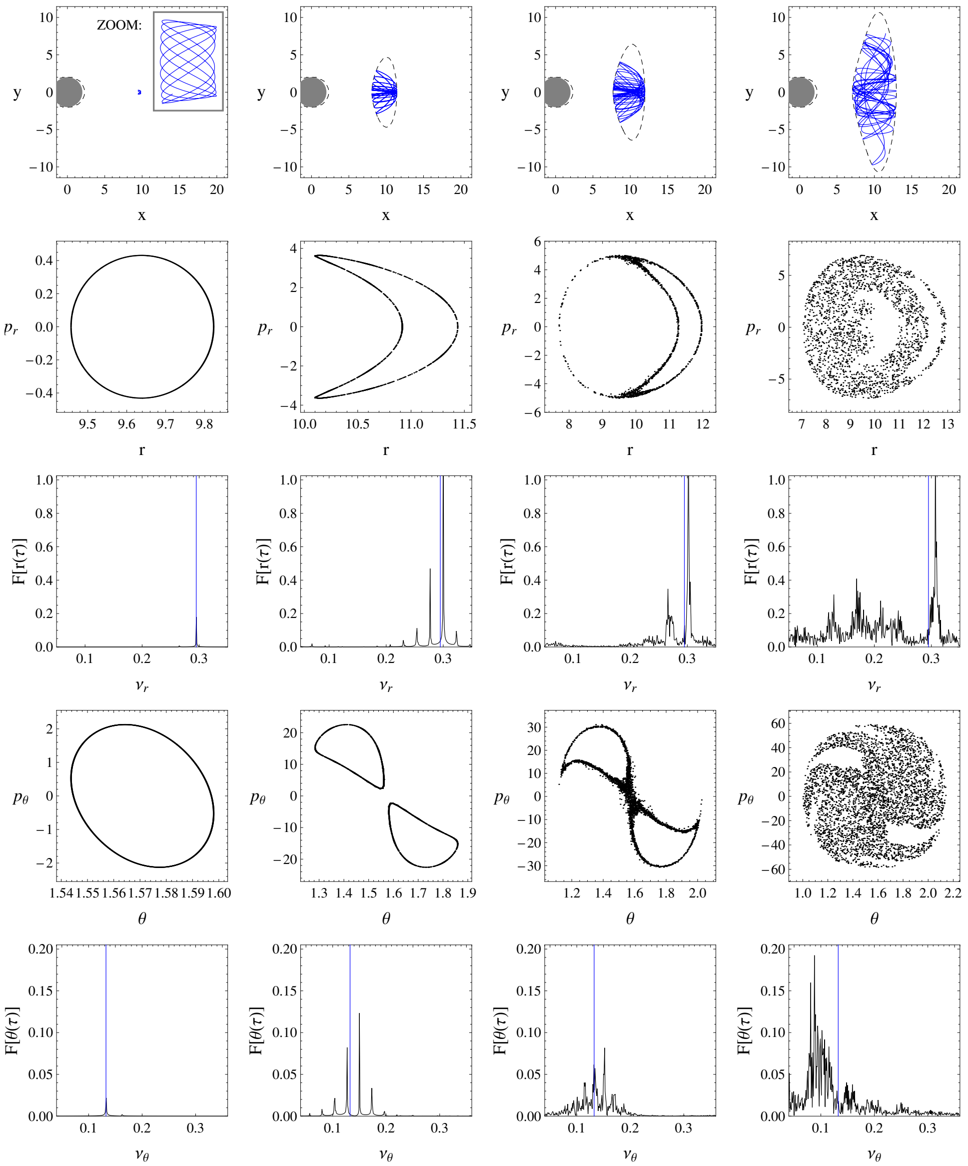
Figure 2. Poincare surface of section r/p_r ($\theta = \pi/2$) for string loop trajectories in the neighbourhood of minima of $E_b(r, \theta)$ function. Resonant (3:2, 1:1, 2:3, 1:2) and nonresonant (1:φ, $r_0 = 9$) radii for $\Omega_\theta(r) : \Omega_r(r)$ frequency ratios are depicted. Every picture contains multiple trajectories and every (regular) trajectory is forming a ring. Trajectories are differing in initial conditions r, P_r, P_θ, but has the same energy E and parameter J. They are bounded by the E_b function, see thick curve. We see destruction of the initial tori for 1:1 and 1:2 and formation of new ones on $P_r = 0$ line. For another resonances and also for nonresonant radii, the initial tori are preserved. The most resilient tori exist for golden frequency ratio 1:φ.

It is quite interesting that the latitudinal frequency of the string loop oscillations in the Schwarzschild or other spherically symmetric spacetimes equals to the latitudinal frequency of the epicyclic geodetical motion as observed by distant observers – for details see (Stuchlík and Kološ, 2012b).

The radial profiles of the string loop oscillations qualitatively differ from those related to the radial oscillations of the geodesic, test particle motion in the Schwarzschild geometry, especially there is a crossing point of the radial and vertical frequencies in the Kerr black hole spacetimes for the string loop oscillation, while for the test particle oscillations such a crossing is possible only in the Kerr naked singularity spacetimes, (Török and Stuchlík, 2005; Stuchlík and Schee, 2012).

4 TRANSITION FROM REGULAR TO CHAOTIC MOTION

According to the Kolmogorov–Arnold–Moser (KAM) theory (Arnold, 1978), a string loop will oscillate in a regular quasi-periodic motion, if the parameter ϵ remains small. The trajectory of such regular motion, restricted by energy (8) in its phase space r, θ, P_r, P_θ,



(a) $E = 19.8$ (b) $E = 20$ (c) $E = 20.15$ (d) $E = 20.5$

Figure 3. Transition from the regular to the chaotic regime of the string loop motion. The string loop is starting from the rest near the local minimum located (for the string parameter $J = 11$) at $r_0 \doteq 9.64, \theta_0 = \pi/2$, with successively increasing energy E. For every energy level we plotted the string loop trajectory, the Poincare surface sections (r, P_{r}),(θ, P_θ) and the Fourier spectrum for both coordinates r and θ (Ott, 1993). The vertical lines in the Fourier spectra are the frequencies $\omega_r/(2\pi), \omega_\theta/(2\pi)$.

will lies on so called KAM torus. As the parameter ϵ grows, the condition $\epsilon \ll 1$ becomes violated, the nonlinear parts in the Hamiltonian become stronger, and the string loop enters the nonlinear, chaotic regime of its motion.

The Birkhoff theorem, ensuring the existence of a canonical transformation (11) putting a Hamiltonian system into normal form (13) up to a remainder of a given order, is violated, if for our two degrees of freedom (2 DOF) (5)

$$k_1\,\omega_1 + k_2\,\omega_2 = 0\,, \quad k_1 + k_2 < 4\,. \tag{22}$$

So for resonances 1:1, 1:2, 2:1 we can not construct normal forms, with frequencies ω_1, ω_2. It does not mean that at resonant radii the motion in the vicinity of minima will not be regular, we still have regular motion close to the minima of E_{b}, but the former KAM tori are destroyed for 1:1, 1:2, 2:1, see Fig. 2.

Increase of non-linearity and chaoticity of a system moving in vicinity of its local stable equilibrium point is caused by increase of its energy. We demonstrate successive transfer from the purely regular, periodic motion through quasi-periodic motion to purely chaotic motion of a string loop in Fig. 3. The Poincare surface sections in the phase space and the Fourier transforms of the oscillatory motion in the radial and latitudinal direction clearly represent the transfer to the chaotic motion. Of course, in the entering phase of the motion with lowest energy, the string loop motion is fully regular and periodic and is represented by appropriate Lissajousse figures.

It is convenient to represent the transfer to the chaotic system by an appropriate Lyapunov coefficient. The chaotic systems are sensitive to initial conditions and we can follow two string loop trajectories separated at the initial time t_0 by a small phase-space distance d_0. As the system evolves, the two orbits will be separated at an exponential rate if the motion of the string loops is in the chaotic regime. The Lyapunov exponent (Ott, 1993)

$$\lambda_{\mathrm{L}} = \lim_{\substack{d_0 \to 0 \\ t \to \infty}} \left(\frac{1}{t} \ln \left(\frac{d(t)}{d_0} \right) \right) \tag{23}$$

is describing the two orbits separation and hence the measure of chaos. The transition from the regular to the chaotic regime of the string loop motion is clearly visible due to the evolution of the maximal Lyapunov exponent (Ott, 1993) demonstrated in Fig. 4. We clearly see strongly increasing measure of chaos with increasing energy of the moving string loop when some critical energy is crossed. This effect is genuine to the dynamical systems and we observed it also for the string loops in the spherically symmetric braneworld spacetimes, (Stuchlík and Kološ, 2012b).

5 CONCLUSIONS

System will oscillate in a quasi-periodic motion, if the parameter ϵ remains small. As the parameter ϵ grows, the condition $\epsilon \ll 1$ becomes violated, the nonlinear parts in the Hamiltonian become stronger, and we enter the nonlinear, chaotic regime of its motion. Increase of non-linearity of a system moving in vicinity of its local stable equilibrium point (minimum) is caused by increase of its energy. The transition from the regular to the chaotic regime of the motion is the solution to the "focusing" problem of the string loop trajectories discussed in (Jacobson and Sotiriou, 2009).

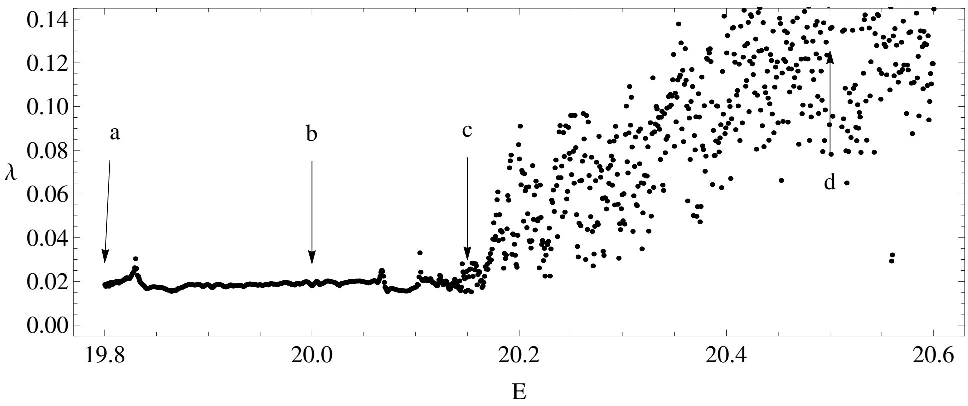


Figure 4. Evolution of the maximal Lyapunov exponent in dependence of on the string loop energy, related to Fig. 3. For small energies the motion is regular, for bigger energies the motion is chaotic – this is manifestation of the KAM theorem. The transition between the regular/chaotic regimes occurs approximately at $E \sim 20.15$. Letters denote the individual cases in Fig. 3.

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