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Pseudo-Newtonian gravitational potential of Schwarzschild black hole in the presence of quintessence

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ABSTRACT
We introduce a pseudo-Newtonian gravitational potential describing the gravitational field of Schwarzschild black hole surrounded by a quintessential field. We also show, how the geodesic motion reflected in behaviour of general relativistic effective potential can be alternatively described by the pseudo-Newtonian one.

Keywords: Schwarzschild black hole – quintessence – geodesic motion – pseudo-Newtonian potential

1 INTRODUCTION

Starting in late seventies, the conception of the so-called pseudo-Newtonian (PN) gravitational potential came up in astrophysics (Abramowicz, 2009). Those days, the observational 'discover' of the black hole Cygnus-X seemed to be widely accepted in astrophysics. Consequently, general relativity started to play its role in investigation of astrophysical processes. Even these days, however, many astrophysicists neglect the effects of general relativity, being focused on processes relatively far from sources of gravity, where the general relativistic effects can be assumed as small corrections to Newtonian calculations only. On the other hand, coming closer to the objects, like compact objects (black holes, neutron stars, etc.) are, the Newtonian calculations lose its validity and general relativity approach must be applied. Accretions discs (toroidal fluid structures) circling round black holes represent the impressive example of this. The accretion disc treated within Newtonian theory does not exhibit the cusp, through which the matter flows onto the black hole. Just the application of general relativistic description shows up the existing cusp (Abramowicz et al., 1978, 1980) Thus, in dependence on studied problems, it is crucial to decide correctly, which approach to apply. The exact and general, but complex general relativistic one, or the approximative, simpler and perhaps more intuitive Newtonian one, but failing in strong gravity very close to compact objects.

In 1980, however, B. Paczyński and P. Wiita introduced the gravitational potential of spherically symmetric static object – the source of strong gravity (e.g. Schwarzschild black hole) $\psi_{\rm PW} = -1/(r - 2GM/c^2)$ in the paper (Paczyński and Wiita, 1980). Being used

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instead of the standard Newtonian one $\psi_{\mathrm{N}} = -GM/r$ in the Newtonian theory, such a gravitational potential 'helps' the Newtonian approach to describe also some features of processes taking place close to Schwarzschild black holes.

There is a variety of different approaches in defining the PN gravitational potential describing different kinds of black holes and various aspects of their spacetime structure (Paczyński and Wiita, 1980; Chakrabarti and Khanna, 1992; Nowak and Wagoner, 1991; Artemova et al., 1996; Semerák and Karas, 1999; Mukhopadhyay, 2002; Mukhopadhyay and Misra, 2003; Ghosh and Mukhopadhyay, 2007; Abramowicz, 2009). In the case of Schwarzschild spacetimes, it seems (Artemova et al., 1996) that to reflect the accretion disc properties, the most convenient is the original Paczyński–Wiita gravitational potential ψ_{PW}. It enables us to calculate positions of the marginally stable and bound circular orbits at the same radii as follow from the general relativistic calculations.

Originally, the Paczyński–Wiita potential was introduced by a guess, when attempting to include the Schwarzschild radius $r = 2GM/c^2$ into the Newtonian gravity. There is, however, a simple heuristic method for derivation of the PN potentials that yields the Paczyński–Wiita potential. The same method was used for the derivation of the PN gravitational potential for the equatorial plane of rotating Kerr black hole as well (Mukhopadhyay, 2002). Then the position of the marginally stable circular orbit corresponds to the position determined by using the general relativistic approach, and differences in positions of marginally bound circular orbit determined in both the ways are relatively small.

Standardly, this kind of approach, i.e. using the common Newtonian routines and formulas, but with the PN gravitational potential is called the PN approach. The gravity, however, is not the only widely manifesting force in the universe influencing the astrophysical processes. Cosmological observations of distant Ia-type supernova explosions indicate an accelerating universe. Starting at the cosmological redshift $z \approx 1$, the accelerated expansion should be generated by some appropriate form of the so-called dark energy (S. Perlmutter *et al.*, 1999; Riess and *et al.*, 2004). These results are in accord with a large variety of cosmological tests including gravitational lensing, galaxy number counts, etc. (Ostriker and Steinhardt, 1995). The recent detailed studies of the cosmic microwave background (CMB) anisotropies indicate that the energy content of the dark energy represents $\sim 74.5\%$ of the energy content in the observable universe, and the sum of energy densities is very close to the critical energy density ρ_{crit}, corresponding to almost flat universe (Spergel D. N. *et al.*, 2003, 2007).

A large variety of possible candidates for the dark energy is discussed these days. First of all, there is the standard possibility represented by the cosmological constant Λ. Its Lorentz invariant form enables interpretation in terms of a ground state or vacuum energy of quantum fields (Dolgov et al., 1988). The energy density ρ_Λ, which can be associated with the cosmological constant, remains unchanged during the cosmic expansion, and its pressure to energy density ratio (equation of state) is $w = p_\Lambda/\rho_\Lambda = -1$.

Further, there is a variety of scalar fields evolving outside of their energy minimum, called quintessence, which possess a time varying energy density and equation of state with $-1 < w < -1/3$ (Zlatev et al., 1999). Such a scenario can be realised by light scalar field coming from modified $f(R)$ gravity (Nojiri and Odintsov, 2003), string-inspired cosmologies (Tsujikawa and Sami, 2001), cosmology with extra dimensions (Neupane, 2004), or by k-essence being a scalar field with a non-canonical kinetic term (Armendariz-

Picon et al., 1999). Similar behaviour is exhibited by the coupled dark energy, i.e. a scalar field coupled to the dark matter. For example, Chaplygin gas or its generalization called quartessence explain both dark energy and dark matter from an unified physical origin (Kamenshchik et al., 2001).

Several years ago, trying to have an effective PN tool even for processes with the dark energy, we constructed the PN gravitational potential describing the gravitational field of Schwarzschild black hole in the universe with the cosmological constant Λ (Stuchlík and Kovář, 2008; Stuchlík et al., 2009). The general relativistic description of such a configuration is represented by the Schwarzschild–de Sitter spacetimes (Stuchlík, 1990; Stuchlík et al., 2000; Stuchlík, 2005). Here we follow this kind of investigation, introducing the PN gravitational potential for the gravitational field of Schwarzschild black hole immersed in a quintessence, representing an alternative explanation of the dark energy.

2 SCHWARSCHILD BLACK HOLE SURROUNDED BY QUINTESSENCE IN GENERAL RELATIVITY

In the standard Schwarzschild coordinates (t, r, θ, ϕ) and the geometric system of units $(c = G = 1)$, the spacetime of Schwarzschild black hole surrounded by a quintessence field is determined by the static and spherically symmetric Kiselev solution of the Einstein equations (Kiselev, 2003)

$$\mathrm{d}s^2 = -g(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{g(r)} + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big)\,, \tag{1}$$

with the lapse function

$$g(r) = 1 - \frac{2M}{r} - \frac{\alpha}{r^{3w+1}}\,, \tag{2}$$

where M is the mass parameter of the spacetime, w is the quintessential state parameter and α is the normalization factor. The quintessential parameter relates the quintessence pressure p and density ρ in the equation of state $p = w\rho$ and takes values from $-1 < w < -1/3$, whereas the limiting value $w = -1$ corresponds to the dark energy not being quintessence but the vacuum energy (cosmological constant). Moreover, in that case of $\alpha = \Lambda/3$, where Λ is the cosmological constant, the solution (1) reduces exactly to the Schwarzschild–de Sitter solution. Comparison of both the solutions is given in the paper (Fernando, 2013).

In the following, we focus on the exemplary case $w = -2/3$, when the metric lapse function takes a simple form

$$g(r) = 1 - \frac{2M}{r} - \alpha r\,. \tag{3}$$

Singularities of the lapse function giving the black-hole and cosmological horizons are determined by the equation $r - 2M - \alpha r^2 = 0$ and are located at

$$r_{\mathrm{bh}} = \frac{1 - \sqrt{1 - 8\alpha M}}{2\alpha}\,, \quad r_{\mathrm{c}} = \frac{1 + \sqrt{1 - 8\alpha M}}{2\alpha}\,. \tag{4}$$

Both the horizons exist for $1 - 8\alpha M > 0$, separating the spacetimes into two dynamic regions and one static region between $r_{\rm bh}$ and $r_{\rm c}$. For $M = 1/(8\alpha)$, both the horizons coalesce at the radius $r_{\rm bh} = r_{\rm c} = 1/(2\alpha)$.

The heuristic method (see, e.g. Mukhopadhyay (2002)), enabling us to define the PN gravitational potential is based on the knowledge of exact general relativistic relations for the angular momentum per particle mass $L_{\rm c}$ and energy per particle mass $E_{\rm c}$ of particles moving along circular geodesics. Then, we have to realize that in Newtonian physics, the Newtonian gravitational potential $\psi_{\rm N}$ for central gravitational fields is related to the Newtonian angular momentum per particle mass $l_{\rm N,c}$ of free particles moving along circular orbits by the relation ${\rm d}\psi_{\rm N}/{\rm d}r = l^2_{\rm N,c}/r^3$. Now, the main idea in definition of the PN gravitational potential ψ is in the transposition $l_{\rm N,c} \rightarrow L_{\rm c}/E_{\rm c} \equiv l_{\rm c}$,[1] thus we define the potential by the relation

$$\psi = \int \frac{L_{\rm c}^2}{E_{\rm c}^2\, r^3}\,{\rm d}r\,. \tag{5}$$

Note that the described method of PN determination works quite well in spherically symmetric (non-rotating) spacetimes, or in the equatorial plane of axially symmetric (rotating, e.g. Kerr or KdS) spacetimes. However, it is much more complicated task to find a PN potential for regions outside the equatorial plane of the rotating spacetimes, because of a non-trivial influence of the dragging of inertial frames. There is a need to upgrade this method (Ghosh and Mukhopadhyay, 2007) or use completely different way of the gravitational potential definition (Semerák and Karas, 1999).

3 CIRCULAR GEODESICS IN GENERAL RELATIVITY

In general relativity, the circular geodesics at $r_{\rm c}$ correspond to extrema of the effective potential, given in the equatorial plane of static and spherically symmetric spacetimes in terms of the metric coefficients $g_{\phi\phi}$ and g_{tt}, and the angular momentum L, by the relation (Misner et al., 1973)

$$V^2_{\rm eff} = -g_{tt}\left(1 + \frac{L^2}{g_{\phi\phi}}\right), \tag{6}$$

thus, for the Schwarzschild-quintessential spacetime, it is given by the relation

$$V^2_{\rm eff} = \left(1 - \frac{2M}{r} - \alpha r\right)\left(1 + \frac{L^2}{r^2}\right). \tag{7}$$

[1] The quantity $l_{\rm c} = L_{\rm c}/E_{\rm c}$ plays its role only when the PN (e.g., Paczyński–Wiita) gravitational potential is defined. Later, standard Newtonian quantities in Newtonian theory are used along with the PN gravitational potential.

The extrema condition for this effective potential, $\partial_r V_{\text{eff}}|_{r_c} = 0$, enables us to determine the constants of motion related to the circular geodesic orbits in the form

$$L_c^2 = \frac{r_c^2(\alpha r_c^2 - 2M)}{\alpha r_c^2 - 2r_c + 6M}\,, \tag{8}$$

$$E_c^2 = -\frac{2(\alpha r_c^2 - r_c + 2M)^2}{r_c(\alpha r_c^2 - 2r_c + 6M)}\,. \tag{9}$$

Dropping now the subscript 'c', the PN gravitational potential (5) can be written in the form

$$\psi = -\frac{r}{2(r - 2M - \alpha r^2)} + \mathcal{K}\,, \tag{10}$$

where $\mathcal{K}$ is an integration constant having no physical meaning, but enabling to specify a proper form of the potential ψ. Here, we demand that for $\alpha = 0$ expression (11) takes the form of the Paczyński–Wiita potential. This corresponds to the choice $\mathcal{K} = 1/2$ and the PN gravitational potential can be then written in its final form

$$\psi = -\frac{2M + \alpha r^2}{2(r - 2M - \alpha r^2)}\,. \tag{11}$$

We can see that the potential diverges at the radii of horizons and reflects the position of the static radius r_s (corresponding to the local maximum of this potential) of the Kiselev spacetime (see Fig. 1).

Along with the horizons, the static radius is the crucial feature of the Kiselev spacetime. It is the radius where the gravitational attraction of the central black hole is balanced by the cosmic repulsion caused by the quintessence matter. In more details, test particle can stay at rest at that radius – its angular momentum must vanish, $L_c = \alpha r_c^2 - 2M = 0$, which determines the static radius as

$$r_s = \sqrt{2M/\alpha}\,. \tag{12}$$

4 TEST-PARTICLE MOTION IN THE PN POTENTIAL

In the case of central gravitational fields, test-particle motion is confined to central planes (e.g. to the equatorial plane). Following the Newtonian physics, the radial equation of the Keplerian equatorial motion can be written in the form

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = e - v_{\text{eff}}\,, \tag{13}$$

where e is the total PN energy per particle mass (energy hereafter) and v_{eff} is the PN effective potential per particle mass (effective potential in the following) defined by the standard relation

$$v_{\text{eff}} = \psi + \frac{l^2}{2r^2}\,. \tag{14}$$

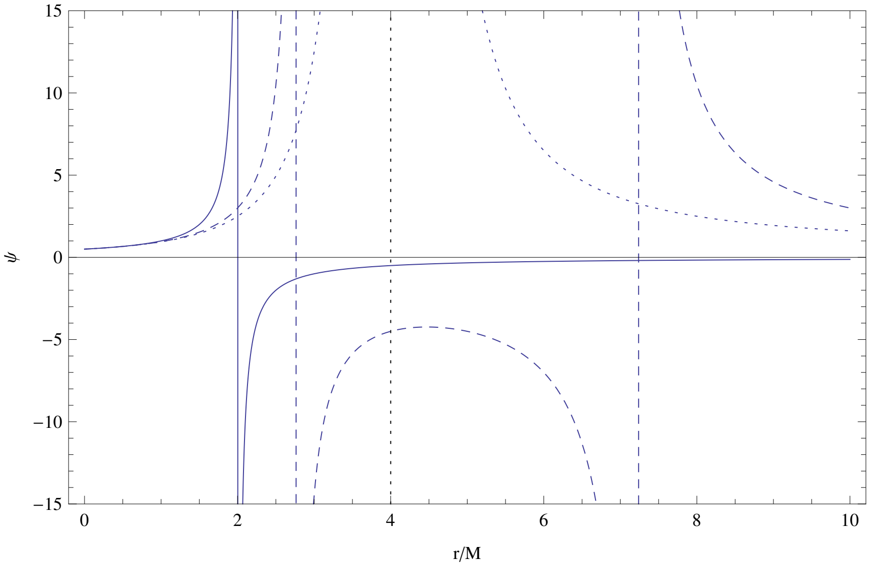

Figure 1. Pseudo-Newtonian gravitational potential for the gravitational field of Schwarzschild black hole surrounded by the quintessence with $w = 2/3$. The solid curve represents the behaviour of the limit case $\alpha = 0$ of the potential, the dashed curve shows its behaviour for $\alpha = 1/10\,\mathrm{M}^{-1}$, while the dotted curve represents the limit case $\alpha = 1/8\,\mathrm{M}^{-1}$, when the horizons coalesce at the radius $r = 4\,\mathrm{M}$.

Here, ψ is the PN gravitational potential (11) and l is the PN angular momentum per particle mass (angular momentum hereafter) defined in Section 2. The circular Keplerian orbits (geodesics) correspond to the effective potential extrema.[2] Thus, their angular momentum is governed by the function

$$l_{\mathrm{c}}^2 = -\frac{r^3\left(\alpha r^2 - 2M\right)}{2\left(\alpha r^2 - r + 2M\right)^2}\,, \tag{15}$$

and the corresponding energy (effective potential extreme) is governed by the function

$$e_{\mathrm{c}} = \frac{2\left(\alpha r^2 + 2M\right)^2 - r\left(3\alpha r^2 + 2M\right)}{4\left(\alpha r^2 - r + 2M\right)^2}\,. \tag{16}$$

[2] Keplerian circular motion can be equivalently given also directly from the PN gravitational potential (11) using relations for orbital and angular velocities, and for the angular momentum and energy

$$v = \left(r\frac{\mathrm{d}\psi}{\mathrm{d}r}\right)^{1/2},\quad \Omega = \left(\frac{1}{r}\frac{\mathrm{d}\psi}{\mathrm{d}r}\right)^{1/2},\quad l_{\mathrm{c}} = \left(r^3\frac{\mathrm{d}\psi}{\mathrm{d}r}\right)^{1/2},\quad e_{\mathrm{c}} = \frac{1}{2}v^2 + \psi\,.$$

However, in some sense, the method of effective potential (Misner et al., 1973), combining the gravitational potential and potential of centrifugal forces, is more general, illustrative and convenient for our case.

5 CONCLUSION

In general, the PN gravitational potential, being used instead of the Newtonian one in the Newtonian approach represents very useful tool to describe the test particle motion (and not only that problem, as we show, e.g. in (Stuchlík et al., 2009)) within the Newtonian physics, taking into account some of the most important features following from general relativity effects when strong gravitational field is present. It also enables us to simply incorporate cosmic repulsive forces (caused by the cosmological constant, quintessence matter, etc.) into our consideration, having an impact on astrophysical phenomena as well.

Few years ago, we presented the construction of the PN gravitational potential for the gravitational field of Schwarzschild black hole in the universe with cosmological constant (Schwarzschild–de Sitter spacetime) and tested its accuracy. We showed that the PN gravitational potential defined for the Schwarzschild–de Sitter spacetimes reflects precisely the existence of the static radius, diverges at both the black-hole and cosmological horizons, and predicts locations of both the inner and outer marginally stable and marginally bound circular orbits at the same radii as those following from the full general relativity (Stuchlík and Kovář, 2008). The energy difference between the inner and outer marginally stable circular orbit, which plays a crucial role in the theory of thin discs, has been shown very close to the relativistic result. We also demonstrated that the PN potential can be well applied even for description of thick discs orbiting Schwarzschild–de Sitter black holes; it provides exact determination of the equipressure (equipotential) surfaces governing the shape of toroidal discs in equilibrium configuration (Stuchlík et al., 2009).

Here, we have presented the construction of the PN gravitational potential describing the gravitational field of a Schwarzschild black hole surrounded by a quintessence representing source of the accelerated expansion of the universe. We have presented its form that reduces to the well-known Paczyński–Wiita gravitational potential (describing the gravitational field of the pure Schwarzschild black hole) when the quintessence parameter α tends to zero (the quintessence is not present). The PN gravitational potential diverges on both the horizons and reflects the position of the static radius as well.

In the future, we plan to deeply go through the testing of accuracy of the presented potential in the same way as we have done for the case of the PN gravitational potential for the Schwarzschild black hole gravitational field and cosmological constant, summarized above.

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