Ejection of string loop from region near black hole horizon

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ABSTRACT
We study ejection speed of current-carrying string-loops governed by the presence of an outer tension barrier and an inner angular momentum barrier in the field of the Schwarzschild black holes. We restrict attention to the axisymmetric motion of string loops with energy high enough, when the string loop can overcome the gravitational attraction and escape to infinity. Due to the chaotic character of the string loop motion, the strings can be scattered and the energy of the string oscillations can be efficiently converted to the energy of the linear motion that can represent a jet motion. We give the condition limiting energy available for conversion onto jet-like motion.

Keywords: string loops – black holes – Schwarzschild – spacetime – accelerating jets

1 INTRODUCTION

Relativistic current carrying strings moving axisymmetrically along the axis of a Kerr black hole (Jacobson and Sotiriou, 2009) or a Schwarzschild-de Sitter black hole (Kološ and Stuchlík, 2010) could in a simplified way represent plasma that exhibits associated string-like behaviour via dynamics of the magnetic field lines in the plasma (Christensson and Hindmarsh, 1999; Semenov et al., 2004) or due to thin isolated flux tubes of plasma that could be described by an one-dimensional string (Spruit, 1981). Tension of such a loop string prevents its expansion beyond some radius, while its worldsheet current introduces an angular momentum barrier preventing the loop from collapsing into the black hole. Such a configuration was also studied in (Larsen, 1994; Frolov and Larsen, 1999). It has been proposed in (Jacobson and Sotiriou, 2009) that this current configuration can be used as a model for jet formation. Here we shall test the possibility to converse motion of a string loop originally oscillating around a black hole in one direction to the perpendicular direction, modelling thus an accelerating jet. It is well known that due to the chaotic character of the motion of string loops, such a transformation of the energy from the oscillatory to the linear mode is possible (Jacobson and Sotiriou, 2009; Kološ and Stuchlík, 2010). Here we make the estimate of efficiency of such a transformation in the Schwarzschild gravitation field.

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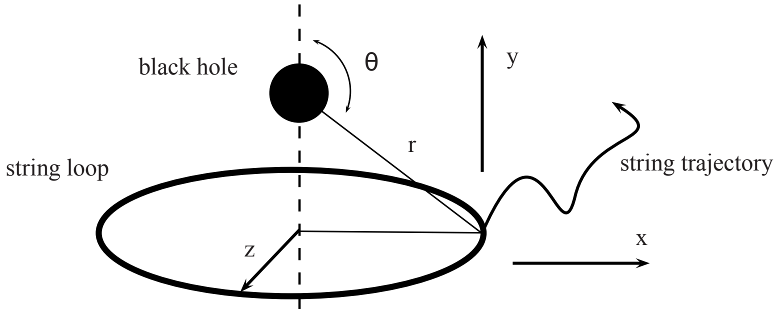

Figure 1. Schematic picture of a string loop moving around a black hole. Assumed axial symmetry of the string loop allows to investigate only one point on the loop; one point path can represent whole string movement. Trajectory of the loop is then represented by the black curve on the picture, given in 2D x-y plot.

2 CURRENT-CARRYING STRING LOOPS

We study a string loop motion in the field of a black hole described by the Schwarzschild metric

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + A^{-1}(r)\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \quad A(r) = 1 - \frac{2M}{r}. \tag{1}$$

We use the geometric units with $c = G = 1$ and the Schwarzschild coordinates. In order to properly describe the string loop motion, it is useful to use the Cartesian coordinates

$$x = r\sin(\theta), \quad y = r\cos(\theta). \tag{2}$$

The string loop is threaded on to an axis of the black hole chosen to be the y-axis – see Fig. 1. The string loop can oscillate, changing its radius in x-z plane, while propagating in y direction. The string loop tension and worldsheet current form barriers governing its dynamics. These barriers are modified by the gravitational attraction of the black hole characterized by the mass M.

2.1 Equations of motion

The string worldsheet is described by the spacetime coordinates $X^\alpha(\sigma^a)$ (with $\alpha = 0, 1, 2, 3$) given as functions of two worldsheet coordinates σ^a (with $a = 0, 1$) that imply induced metric on the worldsheet in the form

$$h_{ab} = g_{\alpha\beta} X^\alpha{}_{,a} X^\beta{}_{,b}. \tag{3}$$

Any two-dimensional metric is conformally flat metric, i.e. in our case when we use the standard Schwarzschild coordinates (1), there is

$$-h_{\tau\tau} = h_{\sigma\sigma} = g_{\phi\phi}, \quad h_{\tau\sigma} = h_{\sigma\tau} = 0, \tag{4}$$

where we adopt coordinates $a = (\tau, \sigma)$.

Dynamics of the string is described by the action related to a scalar field φ and tension μ (worldsheet with minimal area), expressed in the form (Jacobson and Sotiriou, 2009)

$$S = \int \mathrm{d}^2\sigma \sqrt{-h}(\mu + h^{ab}\varphi_{,a}\varphi_{,b}) \,, \tag{5}$$

where $\varphi_{,a} = j_a$ determines current of the string and $\mu > 0$ reflects the string tension. For $j_a = 0$, we have Nambu–Goto string (Zwiebach, 2004), for $j_a = 0, \mu = 0$ we have null string.

Varying the action with respect to φ yields the $1+1$ dimensional wave equation

$$\left(\sqrt{-h}h^{ab}\varphi_{,a}\right)_{,b} = 0 \,. \tag{6}$$

Using the scalar field equation of motion (6) and the assumption of axisymmetry we can conclude that the scalar field can be expressed in linear form with constants j_σ and j_τ

$$\varphi = j_\sigma \sigma + j_\tau \tau \,. \tag{7}$$

Varying the action with respect to the induced metric h_{ab} yields the worldsheet stress-energy tensor density $\tilde{\Sigma}^{ab}$ with the components that can be expressed in the form (Jacobson and Sotiriou, 2009)

$$\tilde{\Sigma}^{\tau\tau} = \frac{J^2}{g_{\phi\phi}} + \mu \,, \quad \tilde{\Sigma}^{\sigma\sigma} = \frac{J^2}{g_{\phi\phi}} - \mu \,, \quad \tilde{\Sigma}^{\sigma\tau} = \frac{-2 j_\tau j_\sigma}{g_{\phi\phi}} \,, \qquad J^2 \equiv j_\sigma^2 + j_\tau^2 \,. \tag{8}$$

Varying the action with respect to X^α, we arrive to equations of motion

$$\left(\tilde{\Sigma}^{ab} g_{\alpha\gamma} X^\alpha_{,a}\right)_{,b} - \frac{1}{2}\tilde{\Sigma}^{ab} g_{\alpha\beta,\gamma} X^\alpha{}_{,a} X^\beta{}_{,b} = 0 \,. \tag{9}$$

In spherically symmetric spacetimes, the axisymmetric string loops can be characterized by coordinates

$$X^\alpha(\tau, \sigma) = \Big(t(\tau), r(\tau), \theta(\tau), \sigma\Big) \,. \tag{10}$$

Equation of motion (9) for coordinates $\gamma = t$ and $\gamma = \phi$ imply two conserved quantities

$$\left(\tilde{\Sigma}^{\tau\tau} g_{tt} \dot{t}\right)_{,\tau} = 0 \,, \quad \left(\tilde{\Sigma}^{\sigma\tau} g_{\phi\phi}\right)_{,\tau} = 0 \,, \tag{11}$$

while for coordinates $\gamma = r$ and $\gamma = \theta$ we obtain two second order ordinary differential equations describing the string motion (Kološ and Stuchlík, 2010).

2.2 Hamiltonian formulation in spherically symmetric spacetimes

The string motion can be also formulated using Hamiltonian formalism (Larsen, 1993). We can consider Hamiltonian

$$\tilde{H} = \frac{1}{2} g^{\alpha\beta} P_\alpha P_\beta + \frac{1}{2}\mu^2 r^2 \sin^2\theta + \mu J^2 + \frac{1}{2}\frac{\left(j_\tau^2 - j_\sigma^2\right)^2}{r^2 \sin^2\theta} \,, \tag{12}$$

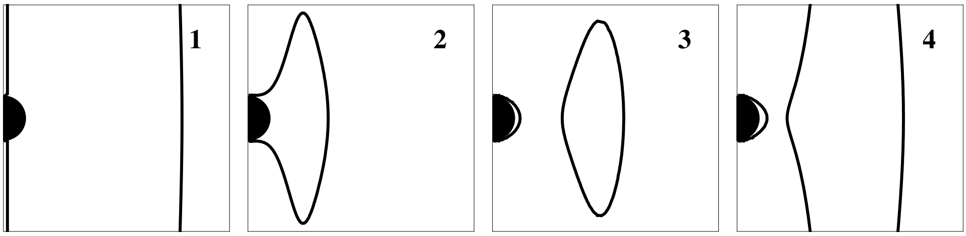

Figure 2. In the Schwarzschild spacetimes, we can distinguish four different types of the behavior of the boundary energy function E_{b}.

where α, β correspond to coordinates t, r, θ, ϕ. The spacetimes symmetries imply existence of two constants of motion

$$P_t = -E\,, \qquad P_\phi = L = -2 j_\tau j_\sigma\,. \tag{13}$$

Then in spherically symmetric spacetimes the Hamiltonian can be expressed in the form

$$H = \frac{1}{2} A(r) P_r^2 + \frac{1}{2}\frac{1}{r^2} P_\theta^2 - \frac{1}{2}\frac{E^2}{A(r)} + \frac{V_{\mathrm{eff}}(r,\theta)}{A(r)}, \tag{14}$$

where an effective potential for the string motion has been introduced by the relation

$$V_{\mathrm{eff}}(r,\theta) = \frac{1}{2} A(r) \left(\mu r \sin\theta + \frac{J^2}{r\sin\theta} \right)^2. \tag{15}$$

The Hamilton equations

$$\frac{\mathrm{d}X^\mu}{\mathrm{d}\lambda} = \frac{\partial H}{\partial P_\mu}, \quad \frac{\mathrm{d}P_\mu}{\mathrm{d}\lambda} = -\frac{\partial H}{\partial X^\mu} \tag{16}$$

applied to the Hamiltonian (14) imply equation of motion in the form

$$r' = A P_r\,, \quad (P_r)' = \frac{1}{A}\frac{P_\theta^2}{r^4}\left(Ar - \frac{1}{2}\frac{\mathrm{d}A}{\mathrm{d}r} r^2 \right) - \frac{\mathrm{d}A}{\mathrm{d}r} P_r^2 - \frac{1}{A}\frac{\mathrm{d}V_{\mathrm{eff}}}{\mathrm{d}r}\,, \tag{17}$$

$$\theta' = \frac{P_\theta}{r^2}\,, \quad (P_\theta)' = -\frac{1}{A}\frac{\mathrm{d}V_{\mathrm{eff}}}{\mathrm{d}\theta}\,. \tag{18}$$

where prime is derivative with respect to the lambda: $f' = \mathrm{d}f/\mathrm{d}\lambda$.

Systems of equations for the string motion in the form (9) and (17–18) are related by transformation

$$\mathrm{d}\tau = \Sigma^{\tau\tau}\,\mathrm{d}\lambda\,. \tag{19}$$

3 STRING LOOP IN SCHWARZSCHILD SPACETIME

The Schwarzschild metric (1) introduces a characteristic length scale corresponding to the radius of the black hole horizon (that is given by the condition $A(r) = 0$, $r_{\mathrm{h}} = 2M$). It is convenient to use the dimensionless coordinates $\tilde{r} = r/M$ ($\tilde{x} = x/M$, $\tilde{y} = y/M$), the dimensionless string (angular momentum) parameter $\tilde{J} = J/M$ and energy $\tilde{E} = E/M$. Then the condition $H = 0$ can be written in the form

$$\tilde{E}^2 = \left((r')^2 + Ar^2(\theta')^2\right) + 2\tilde{V}_{\mathrm{eff}}\,, \tag{20}$$

The conditions $r' = 0$, $\theta' = 0$ determine boundary for the string motion. The boundary energy reads

$$\tilde{E}_{\mathrm{b}}^2 = 2\tilde{V}_{\mathrm{eff}}\,. \tag{21}$$

In Cartesian coordinates it takes the form

$$E_{\mathrm{b}}(x, y) = \sqrt{A(r)}\left(J^2/x + x\mu\right) = \sqrt{A(r)}f(x)\,, \tag{22}$$

where $r = r(x, y) = \sqrt{x^2 + y^2}$. The function $A(r)$ reflects the spacetime properties, while $f(x)$ those of the string loop. The behaviour of the boundary energy function is given by interplay of the functions $A(r)$ and $f(x)$. The local extrema of the boundary energy function E_{b}, given by

$$(E_{\mathrm{b}})'_x = 0 \Leftrightarrow xA'_r f = -2rAf'_x \qquad (E_{\mathrm{b}})'_y = 0 \Leftrightarrow A'_r y = 0\,, \tag{23}$$

are of crucial importance since they determine the regions of different character of the string loop motion.

In the Schwarzschild geometry the extrema equations (23) can be expressed in the form

$$\tilde{J}^2 = \tilde{J}_{\mathrm{E}}^2 \equiv \frac{\tilde{x}^2(\tilde{x}-1)}{\tilde{x}-3}\,, \quad \tilde{y} = 0\,. \tag{24}$$

The boundary energy function E_{b} has two extrema, maximum and minimum, located above the black-hole horizon (at $\tilde{x} > 2$), when

$$\tilde{J} > \tilde{J}_{\mathrm{E(min)}} \doteq 7\,. \tag{25}$$

The detailed discussion of the properties of the effective potential and the string loop motion can be found in (Kološ and Stuchlík, 2010). Here we summarize some relevant results.

We can distinguish four different types of the behaviour of the boundary energy function E_{b} in the Schwarzschild spacetimes; in Fig. 2 we denote them by numbers 1 to 4. The first case corresponds to no inner and outer boundary and the string can be captured by the black hole or escape to infinity. In the second case, there is an outer boundary, the string loop cannot escape to infinity and it must be captured by the black hole. The third case corresponds to the situation when both inner and outer boundary exist and the string is trapped in some region forming a potential “lake” around the black hole. In the fourth case string cannot fall into the black hole but it can escape to infinity (or be trapped).

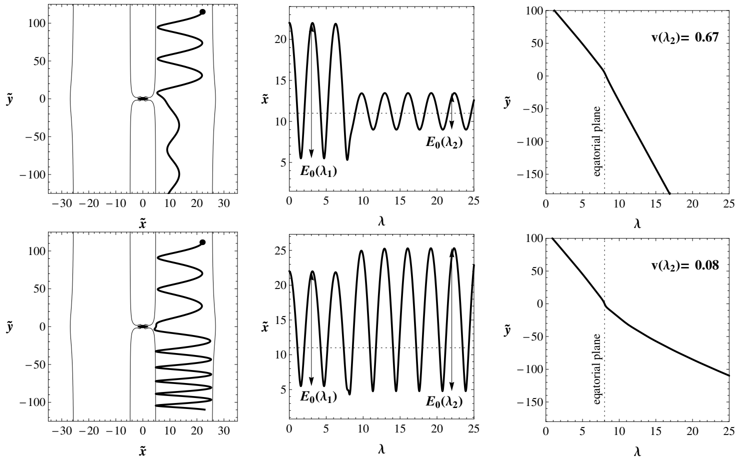

Figure 3. Conversion of energy between E_x and E_y modes – string transmutation effect in the Schwarzschild spacetime. Thick lines represents string trajectory, while thin lines on the first column form boundary for the string motion E_b. The string with current $\tilde{J} = 11$ is starting from region away from black hole horizon $\tilde{x} = 22$ and $\tilde{y} = 115$ (first row), $\tilde{y} = 111.5$ (second row). Near the starting point the spacetime is almost flat, so string oscillates in the x-direction, while moving with initial speed in y direction $v(\lambda_1) \doteq 0.41\,c$ towards the black hole. Around conformal factor $\lambda \sim 8$ the string approaches region near the black hole horizon, where transmutation regime begins, and crosses the equatorial plane. Near the black hole horizon, the modes of energy in the x and y direction are interchanging, and the string is chaotically scattered. First row of pictures represents acceleration of the string in y direction $v(\lambda_2) \doteq 0.67\,c$, while the second one represents its deceleration $v(\lambda_2) \doteq 0.08\,c$.

4 STRING TRANSMUTATION

4.1 Flat spacetime energies

The Schwarzschild metric is flat at infinity. Therefore, we first discuss the motion of the string loop in the flat spacetime. The energy of the string loop (20) in Cartesian coordinates is given by

$$E^2 = (y')^2 + (x')^2 + \left(J^2/x + x\right)^2 = E_\mathrm{y}^2 + E_\mathrm{x}^2\,, \tag{26}$$

where prime is derivative with respect to the affine parameter λ. We introduce energy in x and y directions by the relations

$$E_\mathrm{y}^2 = (y')^2\,, \quad E_\mathrm{x}^2 = (x')^2 + \left(J^2/x + x\right)^2 = (x_\mathrm{i} + x_\mathrm{o})^2 = E_0^2\,. \tag{27}$$

The energy in x direction E_0 (for flat spacetime we introduce new marking $E_\mathrm{x} = E_0$) can be determined by the inner x_i and outer x_o radii limiting motion of the string loop

$$x_\mathrm{o,i} = \frac{1}{2}\left(E \pm \sqrt{E^2 - 4J^2}\right). \tag{28}$$

The energy E_0 is minimal if the inner and the outer radii coincide – then $x_{\rm i} = x_{\rm o} = J$; there is

$$E_{0(\min)} = 2J \, . \tag{29}$$

Clearly, $E_{\rm x}$ and $E_{\rm y}$ are constants of the string loop motion in the flat spacetime and no transmutation is possible.

4.2 Schwarzschild spacetime energies

If the spacetime in not flat, $A(r) \neq 1$, we can write the string loop energy (20) in Cartesian coordinates in the form

$$E^2 = A(r)\Big[g_{xx}\big(x'\big)^2 + 2g_{xy}\big(x'\big)\big(y'\big) + g_{yy}\big(x'\big)^2\Big] + A(r)\big(\tilde{\Sigma}^{\tau\tau}\big)^2 x^2 \, , \tag{30}$$

where metric coefficients for the Schwarzschild spacetime in x and y coordinates are given by

$$g_{xx} = \frac{x^2 + Ay^2}{A\big(x^2+y^2\big)} \, , \quad g_{xx} = \frac{y^2 + Ax^2}{A\big(x^2+y^2\big)} \, , \quad g_{xy} = xy\frac{1+A}{A\big(x^2+y^2\big)} \, . \tag{31}$$

The term $g_{xy}(x')(y')$ is responsible for interchange of energy between $E_{\rm x}$ and $E_{\rm y}$ modes – string transmutation effect. The coefficient g_{xy} is significant only in the neighbourhood of the black hole, so the effect of string transmutation can occur only in this region.

All energy from the $E_{\rm y}$ mode can be transmitted to the $E_{\rm x}$ mode – oscillations of the string loop in the x direction will grown up, while the string will stop moving in the y direction. On the other hand, all energy from $E_{\rm x}$ mode can not be transmitted to the E_y mode – there remains always inconvertible energy $E_{0(\min)} = 2J$ in $E_{\rm x}$ mode, see (29).

The string motion transmutation will change rate of the string propagation in the y direction; an example of acceleration (deceleration) in the y direction can be found on Fig. 3.

4.3 String ejection speed

We consider the toy model of jets represented as string loops starting from region near equatorial plane. We are interested in the maximum speed in y direction that strings can achieve through the transmutation effect, if starting from the rest.

The relativistic gamma factor reads (Jacobson and Sotiriou, 2009)

$$\gamma^2 = \frac{1}{1-v^2} = \frac{E^2}{E_0^2} \, , \tag{32}$$

where E is the energy, E_0 is energy of the string in the x-direction taken at infinity, and v is string velocity in y direction ($v \in [0, 1\,c)$).

Maximal gamma factor (32) (maximal speed) can be obtained if the string loop energy E is large and the final energy in x direction (its value at infinity) is minimal, i.e. $E_{0(\min)} = 2J$.

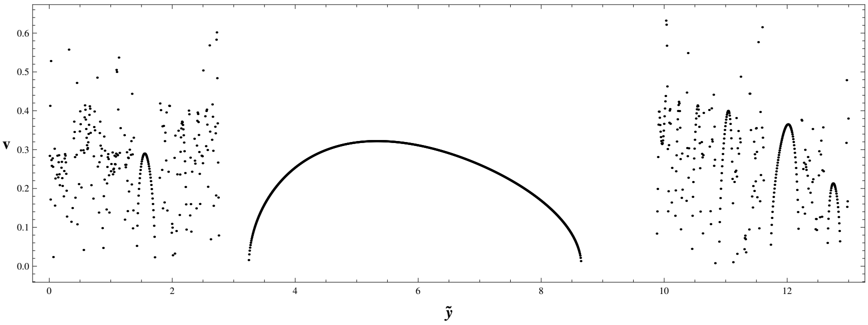

Figure 4. Speed of the string starting from the rest with fixed current $J = 11$ and starting point $\tilde{x} = 20$ while $\tilde{y}$ position (and total energy $\tilde{E}$) is changing. There are velocities up to the $v = 0.65\,c$.

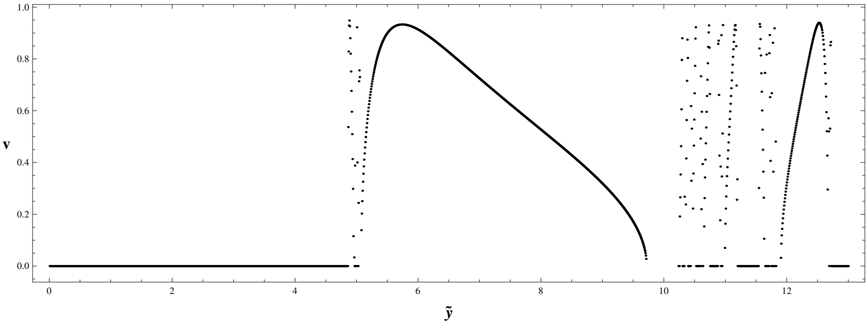

Figure 5. Speed of the string starting from the rest with fixed current $J = 2$ and starting point $\tilde{x} = 20$ while $\tilde{y}$ position (and total energy $\tilde{E}$) is changing. There are velocities up to the $v = c$.

In order to reach acceleration of the string loop in the y-direction the string must past region near the black hole horizon, where string transmutation effect $E_{\rm x} \leftrightarrow E_{\rm y}$ occurs.

Astrophysically most interesting situation corresponds to the string loop initially oscillating in (or near) the equatorial plane when the oscillatory energy is transmitted to the perpendicular direction; such transmutation represents jet ejection. The largest velocities for the string ejection reported in (Jacobson and Sotiriou, 2009) is $v = 0.39\,c$. On the other hand, in our study of strings with $\tilde{J} > \tilde{J}_{\rm E(min)}$ we have found substantially higher values of speed with $v = 0.65\,c$. The results are represented in Fig. 4 clearly demonstrating the chaotic nature of the string transmutation effect. Notice that the regular part of the results of the simulations (in the region $3 < \tilde{y} < 9$) gives maximal $v \sim 0.3\,c$, or $v \sim 0.4\,c$ for $\tilde{y} \sim 11$, in accord with results of (Jacobson and Sotiriou, 2009), while the chaotic region allows $v \sim 0.65\,c$.

There is an important question, whether ultrarelativistic speed of the jet model can be achieved, and under which conditions. The ultra relativistic speeds can be achieved only for small string currents $\tilde{J} < \tilde{J}_{\rm E(min)}$ starting from region out of the equatorial plane. We have type 1 of the motion boundary is such case of $\tilde{J} < \tilde{J}_{\rm E(min)}$ and any string starting close to the equatorial plane will collapse to the black hole. This implies necessity to start the string motion in sufficient distance from the equatorial plane. The results of modelling chaotic string loop motion finishing at infinity for $\tilde{J} < \tilde{J}_{\mathrm{E(min)}}$ is demonstrated in Fig. 5. Notice that now even in the regular part ultrarelativistic speeds $v \sim c$ occur.

5 CONCLUSIONS

We can summarize the possibility of substantial acceleration of string loops that can model jet ejection in the field of Schwarzschild black hole by the statements

- string transmutation effect $E_{\mathrm{x}} \leftrightarrow E_{\mathrm{y}}$ occurs only near the black hole horizon,
- string can be accelerated for $\tilde{J} > \tilde{J}_{\mathrm{min}}$ up to $v \sim 0.65\,c$ and up to $v \sim c$ only for small J and for type 1 of energy boundary.

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