Proceedings of *RAGtime 14–16, 18–22 Sept./15–18 July/11–19 Oct., '12/'13/'14, Opava, Prague, Czech Republic*
Z. Stuchlík, G. Török and T. Pecháček, editors, Silesian University in Opava, 2014, pp. 39–49

Polarization vector transport in Kerr geometry

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ABSTRACT
In this paper we provide a set of practical formulae that are helpful in calculating the orientation of polarization vectors that are parallel transported from a region of strong gravitational field of a Kerr black hole to a distant observer.

Keywords: Polarization – black hole – electromagnetic radiation – synchrotron radiation

1 INTRODUCTION

Polarization of light is an important phenomenon in physics and astronomy makes no exception. When technology allowed astronomers to extend their view of the universe from optical to other spectral bands of electromagnetic spectrum, they also realized that electromagnetic waves carry some more information than just intensity and wavelength. It is the information about polarization of the waves. But it took time until detectors were build able to detect polarization properties of electromagnetic waves.

Polarization of solar radio emission was discovered already in 1939, but was published only in 1946 (Hey, 1946; Reber, 1946). It was followed by discovery of polarization of starlight by Hall (1949) and Hiltner (1949). Radio polarization of Crab nebula was detected in 1954 (Dombrowskii, 1954) and shortly after that Jupiter radiation belt polarization was reported in 1956 (Franklin and Burke, 1956). On the largest scale, polarization of radio galaxies and of the Milky Way was reported in 1962 (Wielebinski et al., 1962; Westerhout et al., 1962).

Highly polarized radiation of synchrotron origin is often observed from active galactic nuclei, where it originates from ejections of jets by super-massive black holes. A super-massive black hole, Sgr A*, rests also in the center of our Galaxy and it is a source of compact non-thermal radio emission (Rogers et al., 1994), which is believed to originate from a synchrotron emitting region closely surrounding the black hole (Beckert and Duschl, 1997; Aitken et al., 2000). Near-infrared observations of Sgr A* from past years revealed repeating simultaneous NIR and X-ray flares of partially polarized (in NIR) emission (e.g. Eckart et al. 2004) that may be produced by synchrotron self-Compton mechanism (Eckart et al., 2008).

978-80-7510-127-3

Models that deal with an emission region that is closer than some $\sim 30\, r_g$ from the central black hole must take into account relativistic effects such as gravitational red-shift and lensing, beaming and light bending, and also the change of polarization angle. For the change of polarization angle, authors usually use the method described by Connors and Stark (1977) and Connors et al. (1980). In this paper, we present a derivation of a simple formula for the change of polarization angle of a linearly polarized synchrotron radiation during parallel transport along the photon path from the point of emission to infinity. The formula is still based on the approach of Connors, Stark and Piran, but hides the details of the parallel transport.

Throughout this paper we use geometrical units with $G = c = M = 1$. According to Misner et al. (1973), we denote 4-vectors either as X^μ or with bold face as $\boldsymbol{X}$ depending on whether we mean their components in a specific basis or we mean an invariant geometrical object independent of coordinate system, respectively. For scalar products of two 4-vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ we use a simplified notation $A^\mu B_\mu = A^\mu B^\nu g_{\mu\nu} = \boldsymbol{A} \cdot \boldsymbol{B} = (A\,B)$.

2 DESCRIPTION OF POLARIZED LIGHT

In the geometrical optics approximation, which is appropriate whenever the wavelength of an electromagnetic wave is much lower than both the typical radius of curvature of the spacetime and the typical length over which wave characteristics like amplitude or wavelength vary, three fundamental assumptions are made: (a) light rays are null geodesics, (b) the number of photons is conserved, and (c) the polarization vector is perpendicular to the rays and is parallel transported along the rays (Misner et al., 1973).

In Maxwell's theory, a monochromatic wave is described by the vector potential

$$\boldsymbol{A} = \Re\left\{\boldsymbol{a}\, e^{i\theta}\right\}, \tag{1}$$

which satisfies the source-free wave equation $\Delta \boldsymbol{A} = 0$ and the Lorentz gauge condition $\nabla \boldsymbol{A} = 0$. Here, $\boldsymbol{a}$ is a slowly varying complex amplitude of the wave and θ is a rapidly varying real phase that is proportional to the distance the wave has travelled and inversely proportional to its wavelength. In general, the amplitude vector $\boldsymbol{a}$ consists of a main part, which is independent of the wavelength λ, plus eventually small corrections that depend on λ and that represent any deviations from pure geometrical optics due to finite wavelength (we will ignore those here).

When seeking for a solution of the wave equation with gauge condition, it is useful to introduce three quantities that describe the electromagnetic wave: wave vector $\boldsymbol{k} = \nabla\theta$, scalar amplitude of the wave $a = (\boldsymbol{a} \cdot \bar{\boldsymbol{a}})^{1/2}$, and polarization vector $\boldsymbol{f} = \boldsymbol{a}/a$. If we then insert the vector potential (1) into the Lorentz gauge condition, we get

$$0 = A^\mu_{\ ;\mu} = \Re\left\{\left[i k_\mu a^\mu + a^\mu_{\ ;\mu}\right] e^{i\theta}\right\} \tag{2}$$

from which we see that $k_\mu a^\mu = 0$ or equivalently that

$$\boldsymbol{k} \cdot \boldsymbol{f} = 0\,, \tag{3}$$

meaning that the polarization vector is perpendicular to the wave vector. From the definition of $\boldsymbol{f}$ we see that it also satisfies

$$\boldsymbol{f} \cdot \boldsymbol{f} = 1 . \tag{4}$$

The polarization state of electromagnetic radiation can be described by a set of Stokes parameters $[I, Q, U, V]$ proposed by Stokes (1852) or alternatively in terms of its total intensity I, (fractional) degree of polarization δ, and the shape parameters of the polarization ellipse ψ and χ.

Both descriptions use total intensity of radiation I as one parameter and the relation between the remaining three Stokes parameters and polarization ellipse parameters is

$$Q/I = \delta \cos 2\psi \cos 2\chi , \tag{5}$$
$$U/I = \delta \sin 2\psi \cos 2\chi , \tag{6}$$
$$V/I = \delta \sin 2\chi , \tag{7}$$

or inversely

$$\delta = \frac{\left(Q^2 + U^2 + V^2\right)^{1/2}}{I} , \tag{8}$$
$$\tan 2\psi = \frac{Q}{U} , \tag{9}$$
$$\sin 2\chi = \frac{V}{\left(Q^2 + U^2 + V^2\right)^{1/2}} , \tag{10}$$

where the factor of two before ψ reflects the rotational symmetry of the ellipse (rotation by 180°) and the same factor before χ reflects another symmetry of 90° rotation and swapping axes.

Stokes' description has the advantage over other ways that Stokes parameters can be expressed in units of spectral density $I(\nu)$, which is what is measured at the end, and that they can be added, which is useful when summing up contributions from many elements of the solid angle.

Most processes in astrophysics produce linearly or highly linearly polarized light. If we only focus on a case of linearly polarized wave ($\chi = 0$), then the polarization ellipse degenerates into a line and instead of two parameters for its description we only need one, which is the polarization angle ψ. The relation between Stokes parameter and polarization ellipse parameters then simplifies into

$$Q/I = \delta \cos 2\psi , \tag{11}$$
$$U/I = \delta \sin 2\psi , \tag{12}$$
$$V/I = 0 \tag{13}$$

and

$$\delta = \left(Q^2 + U^2\right)^{1/2}/I\,, \tag{14}$$

$$\psi = \frac{1}{2}\tan^{-1} Q/U\,. \tag{15}$$

When we denote

$$f_X = Q/I\,, \quad f_Y = U/I\,, \tag{16}$$

then both f_X and f_Y range from -1 to $+1$ and can be viewed as components of a polarization vector $\boldsymbol{f}$ relative to a chosen basis $(\boldsymbol{X}, \boldsymbol{Y})$, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are unit space-like vectors orthogonal to wave-vector $\boldsymbol{k}$. The choice of the polarization plane basis can be arbitrary.

The angle between one of the polarization basis vectors and the polarization vector then defines the polarization angle. While polarization degree is a Lorentz invariant, polarization angle is somewhat loosely defined as it depends on the orientation of an at-will chosen basis.

When a different basis $(\boldsymbol{X}', \boldsymbol{Y}')$ is chosen, which is rotated against the original basis $(\boldsymbol{X}, \boldsymbol{Y})$ by an angle β, the polarization vector (f_X, f_Y) changes to

$$\begin{pmatrix} f'_X \\ f'_Y \end{pmatrix} = \begin{pmatrix} \cos 2\beta & \sin 2\beta \\ -\sin 2\beta & \cos 2\beta \end{pmatrix} \begin{pmatrix} f_X \\ f_Y \end{pmatrix} = \begin{pmatrix} f_X \cos 2\beta & +f_Y \sin 2\beta \\ -f_X \sin 2\beta & +f_Y \cos 2\beta \end{pmatrix} \tag{17}$$

Assume that the original basis is conveniently chosen in such a way that one of its base vectors coincides with the polarization vector ($\boldsymbol{X} = \boldsymbol{f}$, $\boldsymbol{Y} \perp \boldsymbol{X}$). Then $f_Y = 0$ and the above expression simplifies to

$$\boldsymbol{f}' = \left(f_X \cos 2\beta, -f_X \sin 2\beta\right), \tag{18}$$

which after applying some goniometric relations becomes

$$\begin{pmatrix} f'_X \\ f'_Y \end{pmatrix} = \begin{pmatrix} (\boldsymbol{f}\cdot\boldsymbol{X})^2 - (\boldsymbol{f}\cdot\boldsymbol{Y})^2 \\ 2(\boldsymbol{f}\cdot\boldsymbol{X})(\boldsymbol{f}\cdot\boldsymbol{Y}) \end{pmatrix} = \begin{pmatrix} {f_X}^2 - {f_Y}^2 \\ 2 f_X f_Y \end{pmatrix}, \tag{19}$$

where $f_X = \boldsymbol{f} \cdot \boldsymbol{X} = \cos\beta$ and $f_Y = \boldsymbol{f} \cdot \boldsymbol{Y} = \cos(90^\circ - \beta) = \sin\beta$.

Although we started from a special case of conveniently oriented polarization basis, due to its invariant form, the final expression (19) is valid generally.

3 PARALLEL TRANSPORT OF POLARIZATION VECTOR

Kerr spacetime has two obvious symmetries that arise from the fact that it does not depend on time and azimuthal coordinate. This enables one to find two corresponding Killing vectors associated with those differentiable symmetries that satisfy Killing equation $\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0$. The Kerr solution also admits a hidden symmetry represented mathematically by the existence of a Killing tensor field $K'_{\mu\nu}$ – a symmetric tensor field satisfying condition $\nabla_{(\alpha} K'_{\mu\nu)} = 0$ (that the trace-free part of the symmetrization of ∇K vanishes).

According to Noether's theorem (Noether, 1918), all spacetime symmetries are related to conserved quantities. Each Killing vector corresponds to a quantity that is conserved along geodesics, meaning that the product of the Killing vector and the geodesic tangent vector is conserved along the geodesic so that $\mathrm{d}(K_\mu\,\mathrm{d}x^\mu/\mathrm{d}\lambda)/\mathrm{d}\lambda = 0$, where λ is an affine parameter of the geodesic. However, these two constants of motion would not be enough to solve non-equatorial geodesic motion. There exists a third constant of motion, Carter's constant Q (Carter, 1968), which is associated with Killing tensor ($Q = K'_{\mu\nu}u^\mu u^\nu$). Physically, the three constants correspond to the conserved energy, the angular momentum with respect to the symmetric axis of the black hole, and the square of the total angular momentum along the geodesic (Bardeen et al., 1972; Wald, 1984) and enable to solve general geodesic motion in Kerr spacetime (Misner et al., 1973; Chandrasekhar, 1983) analytically.

In addition, Kerr spacetime, as well as other {2, 2} vacuum spacetimes, posses a conformal Killing spinor, which helps to determine parallel propagation of vectors that are perpendicular to geodesics, e.g. 'polarization vectors' (Walker and Penrose, 1970; Chandrasekhar, 1983). Walker and Penrose proved that if $k^\mu(\lambda)$ is a null geodesic and f^μ is a vector such that

$$k^\mu f_\mu = 0 \quad \text{and} \tag{20a}$$

$$f^\mu f_\mu = 1 \tag{20b}$$

(unit vector orthogonal to k^μ; c.f. Eqs. (3) and (4) and parallel propagated along k^μ, then the quantity

$$K_{\mathrm{WP}} = 2\big[\boldsymbol{k}\cdot\boldsymbol{l}\;\boldsymbol{f}\cdot\boldsymbol{n} - \boldsymbol{k}\cdot\boldsymbol{m}\;\boldsymbol{f}\cdot\bar{\boldsymbol{m}}\big]\Psi_2^{-1/3}\,, \tag{21}$$

is conserved along the geodesic, i.e. $k^\mu\nabla_\mu K_{\mathrm{WP}} = 0$. Here l^μ, n^μ, m^μ, $\bar{m}^\mu$ are components of Newmann–Penrose orthonormal tetrad and Ψ_2 is the only non-zero Weyl scalar representing the gravitational monopole of Kerr metric.

Since Ψ_2 is a complex scalar, we can write K_{WP} in the form

$$K_{\mathrm{WP}} = K_1 + i\,K_2 \tag{22}$$

and in Boyer–Lindquist coordinates (see e.g. Walker and Penrose 1970; Connors and Stark 1977; Chandrasekhar 1983; Li et al. 2009 for details) we find that

$$K_1 + i\,K_2 = (A - iB)(r - ia\cos\theta)\,, \tag{23}$$

where

$$A = \big(k^t f^r - k^r f^t\big) + a\sin^2\theta\big(k^r f^\phi - k^\phi f^r\big)\,, \tag{24}$$

$$B = \Big[\big(r^2 + a^2\big)\big(k^\phi f^\theta - f^\phi k^\theta\big) + a\big(k^t f^\theta - k^\theta f^t\big)\Big]\sin\theta\,. \tag{25}$$

If we evaluate K_{WP} at any given point, we can then solve the parallel transport of vector $\boldsymbol{f}$ along the whole geodesic. Equation (23) is equivalent to two real equations plus we have the condition of orthogonality (20a) that does not follow from Walker–Penrose theorem and is an independent one. With these three equations we can fix three space-like components of vector $\boldsymbol{f}$. The time-like component of $\boldsymbol{f}$ can be chosen arbitrarily at a fixed point, because $\boldsymbol{f}$ is defined only up to an additive multiple of $\boldsymbol{k}$. This follows trivially from the fact that $\boldsymbol{k}$

is a null vector and that for $\boldsymbol{f}' = \boldsymbol{f} + \alpha\boldsymbol{k}$ both conditions (20a) and (20b) are satisfied. The polarization vector can be thus written without loss of generality, e.g. as

$$f^{\mu} = \left(0, \cos\psi\, X^{i} + \cos\psi\, Y^{i}\right), \tag{26}$$

for some polarization basis characterized by space-like vectors $(\boldsymbol{X}, \boldsymbol{Y})$ orthogonal to $\boldsymbol{k}$.

When the above choice of f^{μ} is plugged into Eq. (23) and the set of equations is solved for a null geodesic that passes through a point (t, r, θ, ϕ), it is possible to express the final polarization vector at infinity in terms of photon's constants of motions λ and Q. The relative change of polarization angle due to parallel transport is then (Connors et al., 1980)

$$\Delta\chi = \tan^{-1}\left(\frac{-S\,K_2 + T\,K_1}{-S\,K_1 - T\,K_2}\right), \tag{27}$$

where

$$S = \frac{\lambda}{\sin\theta_{\rm obs}} - a\sin\theta_{\rm obs} = -\alpha - a\sin\theta_{\rm obs}, \tag{28}$$

$$T = \mathrm{sgn}\big(k^{\theta}\big)_{\infty}\big(Q - \lambda^2\cot^2\theta_{\rm obs} + a^2\cos^2\theta_{\rm obs}\big)^{1/2} = \beta \tag{29}$$

with a, α, β and $\theta_{\rm obs}$ being respectively the black-hole spin, the horizontal and vertical impact parameters of the null geodesic on the observer's image plane and the observer's inclination angle.

4 APPLICATION TO SYNCHROTRON RADIATION

4.1 Synchrotron emission

When high-energy charged particles (especially electrons) move fast through magnetic fields while they move along magnetic field lines), synchrotron radiation is produced. Synchrotron radiation is like standard cyclotron radiation with the difference that the energetic particles have relativistic speeds and the observed frequency of radiation is affected by the Doppler effect and by the Lorentz factor γ. Another factor γ comes from the relativistic length contraction, which can put the radiation spectrum into the X-ray range. The radiated power is given by the relativistic Larmor formula while the force on the emitting electron is given by the Abraham–Lorentz–Dirac force.

Two main characteristics of astronomical synchrotron radiation include non-thermal power-law spectra and polarization. Following Rybicki and Lightman (1979), a power-law distribution of electrons $n(E)\,\mathrm{d}E \sim E^{-p}\,\mathrm{d}E$ has the specific intensity distribution $I(\nu) \sim \nu^{-s}$, where $s = (p-1)/2$ is the spectral index and maximal degree of polarization

$$\delta = \frac{s+1}{s+5/3} = \frac{p+1}{p+7/3}. \tag{30}$$

The simplified model of local synchrotron emissivity (taking into account only the power-law part of the synchrotron spectrum) is then gives spectral density

$$I(\nu) = I_0\left(\frac{\rho}{\rho_0}\right)\left(\frac{B}{B_0}\sin\vartheta\right)^{1+s}\nu^{-s}, \tag{31}$$

where ϑ is the local angle between the magnetic field and the direction of emission and the relation is valid up to the critical frequency $\nu_{\rm c}$.

4.2 Polarization vector for linearly polarized radiation

In this section we derive practical formulae for determining initial polarization vector $\boldsymbol{f}$ at a point of emission.

Let us assume a packet of synchrotron radiation emitted from a certain place by electrons that pass this place moving along magnetic field lines of intensity $\boldsymbol{B}$. Photons from the packet that eventually reach an observer at infinity all have 4-momentum $\boldsymbol{P}$ The electrons have some local distribution of energies and velocities, but they are part of a fluid with bulk motion characterized by 4-velocity $\boldsymbol{U}$.

From the properties of synchrotron radiation we know that the radiation will be partially linearly polarized and that the direction of polarization will be perpendicular to the projection of magnetic field onto the polarization plane.

The polarization plane contains polarization vector $\boldsymbol{f}$ (it is the plane in which electric and magnetic field vectors oscillate) and it is perpendicular to the direction of photon propagation. We can determine the polarization plane by constructing its normal vector $\boldsymbol{Z}$ pointing in the direction of propagation of the wave. From the the concept of ideal magneto-hydrodynamics (perfectly conducting fluid) it follows that $\boldsymbol{B}$ is a space vector ($\boldsymbol{B} \cdot \boldsymbol{U} = 0$) and so $\boldsymbol{Z}$ and $\boldsymbol{f}$ must be space-vectors too.

Because the direction of propagation is given by photons' 4-momentum $\boldsymbol{P}$, we are looking for a unit vector in a form

$$\boldsymbol{Z} \sim \boldsymbol{P} + \alpha_1 \boldsymbol{U} \,, \tag{32}$$

that has to satisfy $\boldsymbol{Z} \cdot \boldsymbol{Z} = 1$ (unit vector) and $\boldsymbol{Z} \cdot \boldsymbol{U} = 0$ (space vector). It is easy to find that

$$\boldsymbol{Z} = \boldsymbol{U} + \frac{\boldsymbol{U}}{(U\,P)} \,. \tag{33}$$

Now, with the help of $\boldsymbol{Z}$, we can find a projection of magnetic field vector $\boldsymbol{B}$ onto the polarization plane simply as

$$\boldsymbol{B}_{\perp} = \boldsymbol{B} - \alpha_2 \boldsymbol{Z} \,, \tag{34}$$

where $\alpha_2 \boldsymbol{Z} = \boldsymbol{B}_{\parallel}$ stands for the component of $\boldsymbol{B}$ parallel with the direction of propagation. The factor α_2 can be fixed from the condition $\boldsymbol{B} \cdot \boldsymbol{P} = 0$ and so we have

$$\boldsymbol{B}_{\perp} = \boldsymbol{B} - \frac{(B\,P)}{(U\,P)} \boldsymbol{Z} \,. \tag{35}$$

Our polarization vector $\boldsymbol{f}$ can be then chosen in the direction of $\boldsymbol{B}_{\perp}$.

$$\boldsymbol{f} = \frac{\boldsymbol{B}_{\perp}}{|\boldsymbol{B}_{\perp}|} = \frac{(U\,P)\,\boldsymbol{B} - (P\,B)\,\boldsymbol{Z}}{\left[(U\,P)^2\,B^2 - (P\,B)^2\right]^{1/2}} \,. \tag{36}$$

4.3 Linear polarization at infinity

For a practical calculation of parallel transport of polarization vector, we are going to need to define a set of base vectors in the polarization plane.

We have free will in setting up the polarization plane basis and that can be done in a number of ways. Since one usually constructs an orthonormal basis that defines a local reference frame of the fluid (LRF), we may use that frame for construction of our polarization basis. Because LRF is a Cartesian frame by definition, we may choose a completely random vector $\tilde{\boldsymbol{r}}$, from which we derive two other vectors $\boldsymbol{V}$ and $\boldsymbol{W}$ that are both perpendicular to $\boldsymbol{P}$ and they are also perpendicular one another.

$$\boldsymbol{V} = \tilde{\boldsymbol{r}} \times \boldsymbol{P} \quad \text{and} \quad \boldsymbol{W} = \boldsymbol{V} \times \boldsymbol{P} \,. \tag{37}$$

The choice of the initial random vector $\tilde{\boldsymbol{r}}$ can be simply made as $\tilde{r}^{(a)} = (0, r_1, r_2, r_3)$ in LRF, where r_1, r_2, r_3 are non-zero independent random numbers (the time component is zero for the vector to be a space vector). Due to the fact that scalar product of vectors is invariant, vectors $\boldsymbol{V}$, $\boldsymbol{W}$ and $\boldsymbol{P}$ remain perpendicular in any frame of reference.

We must still make sure that the base vectors of our polarization plane are unit vectors and that they lie in the plane. For that reason we introduce another set of vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$, where

$$\boldsymbol{X} = \boldsymbol{V} + \alpha_3 \boldsymbol{P} \,, \tag{38a}$$

$$\boldsymbol{Y} = \boldsymbol{W} + \alpha_4 \boldsymbol{P} \tag{38b}$$

and we require that $|\boldsymbol{X}| = |\boldsymbol{Y}| = 1$ and that $\boldsymbol{X} \cdot \boldsymbol{U} = \boldsymbol{Y} \cdot \boldsymbol{U} = 0$. The later condition allows us to fix α_3 and α_4 and we have

$$\boldsymbol{X} = \left(\boldsymbol{V} - \frac{(V\,U)}{(P\,U)}\, \boldsymbol{P} \right) \frac{1}{(V\,V)^{1/2}} \,, \tag{39a}$$

$$\boldsymbol{Y} = \left(\boldsymbol{W} - \frac{(W\,U)}{(P\,U)}\, \boldsymbol{P} \right) \frac{1}{(W\,W)^{1/2}} \,. \tag{39b}$$

These are the base vectors of our polarization plane. It is easy to verify that $\boldsymbol{X} \cdot \boldsymbol{Y} = 0$.

The last thing we need to do is to project polarization vector $\boldsymbol{f}$ into our polarization plane basis:

$$f_X = \boldsymbol{f} \cdot \boldsymbol{X} = \frac{(U\,P)(B\,V) - (B\,P)(U\,V)}{\left[(U\,P)B^2 - (P\,B)^2\right]^{1/2} (V\,V)^{1/2}} \,, \tag{40a}$$

$$f_Y = \boldsymbol{f} \cdot \boldsymbol{Y} = \frac{(U\,P)(B\,W) - (B\,P)(U\,W)}{\left[(U\,P)B^2 - (P\,B)^2\right]^{1/2} (W\,W)^{1/2}} \,. \tag{40b}$$

Coming back to Eq. (16) and (19), we can express the angle between polarization plane and our polarization basis $(\boldsymbol{X}, \boldsymbol{Y})$ (the polarization angle) as

$$\chi = \frac{1}{2} \tan^{-1} \left(\frac{2 f_X f_Y}{f_Y^2 - f_X^2} \right) . \tag{41}$$

and its change due to strong gravity effects from Eq. (27) as

$$\Delta\chi = \tan^{-1}\left(\frac{-S\,K_2 + T\,K_1}{-S\,K_1 - T\,K_2}\right) . \tag{42}$$

The final set of Stokes parameters for a single light-ray is then

$$I(\nu) = g^3\,I_0(\nu/g)\,, \tag{43a}$$
$$Q(\nu) = I(\nu)\,\delta\,\cos(2\chi + 2\Delta\chi)\,, \tag{43b}$$
$$U(\nu) = I(\nu)\,\delta\,\sin(2\chi + 2\Delta\chi)\,, \tag{43c}$$
$$V(\nu) = 0\,, \tag{43d}$$

where g is the relativistic red-shift factor relating the place of emission and infinity, δ is the conserved polarization degree and I_0 is the specific intensity of the emission (see Eqs. 30 and 31).

5 SUMMARY

We have presented a detailed description of the theory of parallel transport of polarized light in Kerr spacetime with an emphasis on the example of parallel transport of linearly polarized light.

In case of linear polarization induced by synchrotron radiation produced in a fluid with a magnetic field, we give useful and practical formulae for numerical computation of the observed polarization angle. The procedure can be summarized to a step-by-step guide of how to proceed:

(1) At a given point (t, r, θ, ϕ) evaluate the vector of magnetic field $\boldsymbol{B}$, 4-velocity $\boldsymbol{U}$ of the fluid, and the photon 4-momentum vector $\boldsymbol{P}$.

(2) Set up an orthonormal tetrad for the local rest frame (LRF) based on $\boldsymbol{U}$ and in LRF construct a random vector $\tilde{\boldsymbol{r}}$ and from that two vectors $\boldsymbol{V}$ and $\boldsymbol{W}$ that are perpendicular to each other and to the direction of $\boldsymbol{P}$ (Eq. 37).

(3) Calculate the necessary scalar products ($\boldsymbol{U}\cdot\boldsymbol{P}$, $\boldsymbol{B}\cdot\boldsymbol{P}$, $\boldsymbol{B}\cdot\boldsymbol{B}$ and $\boldsymbol{U}\cdot\boldsymbol{V}$, $\boldsymbol{U}\cdot\boldsymbol{W}$, $\boldsymbol{V}\cdot\boldsymbol{V}$, $\boldsymbol{W}\cdot\boldsymbol{W}$) and with the help of those evaluate the components f_X, f_Y of the polarization vector (Equation 40).

(4) Evaluate the polarization angle and its change due to parallel transport using Eqs. (41) and (42).

(5) Integrate the resulting Stokes parameters at the detector using Eq. (43).

ACKNOWLEDGEMENTS

The present work was supported by the following projects: RVO:67985815 (MB,JH) and EC-FP7 StrongGravity 312789 (MB, GK). We thank Michal Dovčiak for useful discussion and comments during the work.

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