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Frequency spectrum of axisymmetric horizontal oscillations in accretion disks

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ABSTRACT
We present the spectrum of eigenfrequencies of axisymmetric acoustic-inertial oscillations of thin accretion disks for a Schwarzschild black hole modelled with a pseudo-potential. There are nine discrete frequencies, corresponding to trapped modes. Eigenmodes with nine or more radial nodes in the inner disk belong to the continuum, whose frequency range starts somewhat below the maximum value of the radial epicyclic frequency. The results are derived under the assumption that the oscillatory motion is parallel to the midplane of the disk.

Keywords: Relativistic stars: black holes – structure stability – oscillations – relativity – gravitation – accretion disks – hydrodynamics

1 ACOUSTIC-INERTIAL MODES

We consider acoustic-inertial modes of oscillation in the inner part of an accretion disk, closely following the formalism of Nowak and Wagoner (1991, 1992). Trapping of the fundamental axisymmetric mode with no nodes in the vertical (z) direction was first demonstrated by Kato and Fukue (1980) in the Schwarzschild geometry. Nowak and Wagoner (1991) derive the equations of motion in a Lagrangian pseudo-Newtonian formalism and specialize to purely horizontal perturbed motions of the disk deriving eigenmodes and eigenfrequencies for the $m = 0$ (axisymmetric) and $m = 2$ (quadrupole) modes. Khanna et al. (2014) computed in an improved pseudo-potential the lowest radial modes (with up to three radial nodes) for azimuthal numbers $m = 0$ through $m = 4$. Here, we present the complete spectrum of horizontal axisymmetric acoustic-inertial disk modes in a pseudopotential which reproduces the properties of the Schwarzschild-metric epicyclic frequency (Kluźniak and Lee, 2002; Khanna et al., 2014). The eigenfrequencies could be related to the quasi-coherent frequencies (QPOs) observed in the X-ray flux from black hole and neutron star systems (for a review see van der Klis M., 2000), as well as in cataclysmic variables (Woudt and Warner, 2002, and references therein).

2 EQUATION OF MOTION AND THE BOUNDARY CONDITION

We model the Schwarzschild metric with a Newtonian pseudo-potential that reproduces the Schwarzschild ratio of $\kappa^2(r)/\Omega^2(r) = 1 - 6GM/(rc^2)$:

$$\Phi_{\rm KL}(r) = -\left(c^2/6\right) \exp\left(\frac{6GM}{rc^2} - 1\right) . \qquad (1)$$

We have dropped an additive constant and renormalized the original Kluźniak and Lee (2002) potential by a factor of $1/e$ to guarantee the correct Schwarzschild value of $\Omega(r_{\rm ms})$. The orbital frequency can be obtained from $\Omega^2(r) = r^{-1}\partial\Phi_{\rm KL}/\partial r$, the radial epicyclic frequency from $\kappa^2 = (2\Omega/r){\rm d}(r^2\Omega)/{\rm d}r$ and the marginally stable orbit is at the zero of κ, at $r_{\rm ms} = 6GM/(rc^2)$. Figure 1 compares our $\kappa^2(r)$ with the Schwarzschild form and two other well-known pseudo-Newtonian models (Paczyński and Wiita, 1980; Nowak and Wagoner, 1991).

In this contribution we assume axisymmetric ($m = 0$) horizontal modes, with the perturbation vector in cylindrical coordinates $(\xi^r_*, \xi^\phi_*, \xi^z_*) = (\xi^r, \xi^\phi, 0)\exp(i\sigma t)$. We use the equation of motion for $\Psi(r) \equiv \sqrt{\gamma Pr}\,\xi^r(r)$ derived in the Lagrangian formalism of Friedman and Schutz (1978) by Nowak and Wagoner (1991)

$$c_{\rm s}^2{\rm d}^2\Psi/{\rm d}r^2 + \left(\sigma^2 - \kappa^2\right)\Psi = 0\,,$$

who also show that in the WKB approximation the azimuthal component of the equation of perturbed motion for thin disks reduces to $\xi^\phi = 2\,i(\Omega/\sigma)\xi^r$.

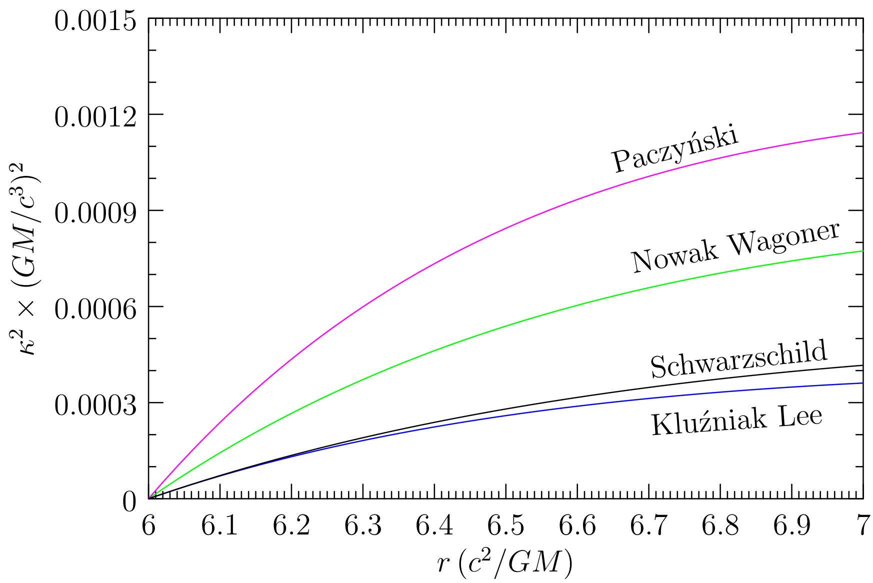

Figure 1. The Schwarzschild epicyclic frequency (squared) and its Newtonian models, from top to bottom: Paczyński and Wiita (1980); Nowak and Wagoner (1991), our Eq. (1).

Following Khanna et al. (2014) we rewrite the equation of motion, and the boundary condition that the Lagrangian perturbation of pressure vanishes at the unperturbed boundary, in dimensionless form as

$$\frac{d^2\Psi}{dx^2} + \left(\frac{a}{H}\right)^2 \left(\tilde{\sigma}^2 - \tilde{\kappa}^2\right)\Psi = 0\,, \tag{2}$$

with the boundary condition at $x = 0$

$$\frac{d\Psi}{dx} = -\frac{\Psi}{2}\,, \tag{3}$$

where H is the half-thickness of the disk, $a = r_{\rm ms}$, and the dimensionless variables are given by $r = a(1+x)$, $\tilde{\sigma} = \sigma/\Omega(a)$, $\tilde{\kappa}(x) = \kappa(r)/\Omega(a)$. The speed of sound $c_{\rm s} = \sqrt{\gamma P/\rho}$ was eliminated with the condition of vertical hydrostatic equilibrium.

3 THE EIGENFREQUENCY SPECTRUM

We have numerically solved the eigenvalue problem given by Eqs. (2) and (3), for a thin disk of $H/a = 10^{-3}$, and present in Table 1 the eigenfrequencies for modes with $\mu = 0, 1, \ldots, 9$ radial nodes in the inner disk. The lowest nine eigenfrequencies ($\mu = 0$ through 8), exhausting the discrete spectrum, correspond to oscillations which are trapped in the inner disk. As already noted by Kato and Fukue (1980), for eigenfrequencies exceeding the maximum of the epicyclic frequency, $\sigma^2 > \kappa^2_{\rm max}$, the acoustic wave ranges throughout the disk (see also Kato et al., 1998), these frequencies belong to the continuum spectrum. The tenth entry in Table 1, with $\mu = 9$ radial nodes in the inner disk, also belongs to the continuum, although it has a frequency below the maximum of the epicyclic frequency $\sigma^2 < \kappa^2_{\rm max}$ (Figs. 2, 3).

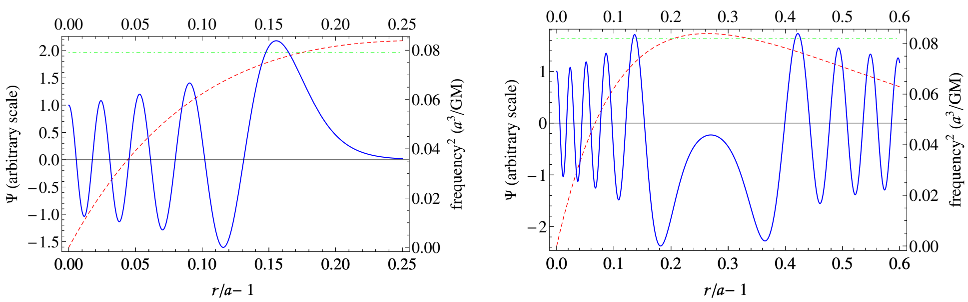

Figure 2. Two radial overtones for $m = 0$ horizontal oscillations of a thin ($H/a = 10^{-3}$) accretion disk. *Left Panel:* A trapped oscillation with $\mu = 8$ radial nodes. *Right Panel:* An oscillation penetrating the epicyclic barrier (with $\mu = 9$ radial nodes in the inner accretion disk and an unlimited number of radial nodes in the outer disk). Plotted are the radial wavefunction $\Psi \propto r^{1/2}\xi^r$: solid blue line (left scale); eigenfrequency (squared) and the epicyclic frequency (squared), both normalized to orbital frequency at the inner edge of the disk, $\sigma^2/\Omega^2(r_{\rm ms})$: dashed-dotted green line, $\kappa^2(r)/\Omega^2(r_{\rm ms})$: dashed red line (right scale).

Table 1. Spectrum of eigenfrequencies for horizontal modes

mode m	radial nodes μ	eigenfrequency $\sigma \times \sqrt{(a^3/GM)}$	mode status	ref.
0	0	0.0988290000	trapped	Khanna et al. (2014)
0	1	0.1721372560	trapped	Khanna et al. (2014)
0	2	0.2049091375	trapped	Khanna et al. (2014)
0	3	0.2269122200	trapped	Khanna et al. (2014)
0	4	0.2433069000	trapped	this work
0	5	0.2561130000	trapped	this work
0	6	0.2663460000	trapped	this work
0	7	0.2745830000	trapped	this work
0	8	0.2811589000	trapped	this work
0	9	0.2862229000	not trapped	this work

In the Figure 2 we present the two eigenmodes corresponding to the last two entries in Table 1. The equations being linear in Ψ, we normalize the wavefunction to unity at the inner edge of the disk: $\Psi(r_{\rm ms}) = 1$ for illustration purposes. The left panel shows the highest-frequency eigenmode in the discrete portion of the spectrum of axisymmetric ($m = 0$) horizontal disk oscillations, the wavefunction of this mode has $\mu = 8$ radial nodes. Note that the wave becomes evanescent for $\sigma^2 < \kappa^2$, thus trapping the $\mu = 8$ mode to the left of $\kappa_{\rm max}$. Some of the lower overtones have been illustrated in Khanna et al. (2014).

The right panel of Fig. 2 illustrates one of the lowest frequency modes in the continuum. Here, $\sigma < \kappa_{\rm max}$ and is so close in value to $\kappa_{\rm max}$ that the wave is transmitted through the epicyclic barrier to the outer disk, where it has an unlimited number of radial nodes in addition to the $\mu = 9$ radial nodes in the inner disk. As far as we are aware, this is a new finding, which has never been reported before. It may have an interesting astrophysical consequence. If the oscillations arise close to the marginally stable orbit, as suggested by Paczyński (1987), the ones transmitted to the outer disk are likely to be more easily observable, in that they may modulate the emission from large parts of the disk.

4 DISCUSSION

We consider accretion disk oscillations in a Newtonian model of the Schwarzschild metric, Eq. (1), which accurately models the radial epicyclic frequency, at least close to the marginally stable orbit, see Fig. 1. No model is perfect, so although we correctly reproduce the ratio of epicyclic to orbital frequency $\kappa(r)/\Omega(r) = \sqrt{1 - 6GM/(rc^2)}$, and the correct value of orbital frequency at the marginally stable orbit, $\Omega(r_{\rm ms}) = c^3/(\sqrt{216}\,GM)$, the maximum of κ occurs at $r = (3 + \sqrt{21})(GM/c^2) \approx 7.58(GM/c^2)$ instead of the Schwarzschild value $r = 8(GM/c^2)$. Further, the equations of motion for the oscillation of the disk fluid were derived in a Newtonian formalism, not in full GR. These departures from GR may limit the quantitative accuracy of the presented results when applied to real

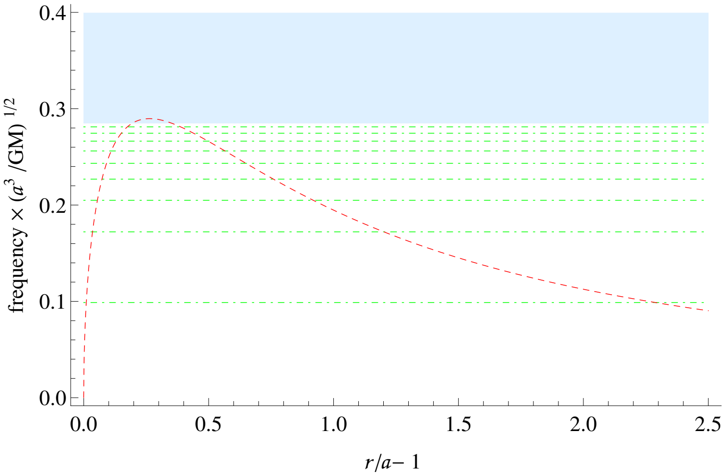

Figure 3. The spectrum of $m = 0$ (axially symmetric) horizontal oscillations of a thin ($H/a = 10^{-3}$) accretion disk for the potential of Eq. (1). Plotted are the epicyclic frequency (*dashed red curve*), and the eigenfrequencies σ in the discrete set (*dashed-dotted green lines*) and in the continuum (*shaded blue region*). All frequencies were scaled with $(GM/a^3)^{1/2}$. Here, and throughout the paper, $a = r_{\rm ms}$.

black hole (or neutron star) accretion disks. An additional assumption which may not be quite accurate is that the oscillations of the disk are strictly parallel to the midplane of the disk, i.e. that the perturbation vector has a zero vertical component, $\xi^z = 0$.

We find that the spectrum of horizontal oscillations is composed of nine discrete frequencies and a continuum (Fig. 3). For the discrete spectrum the wave propagation region corresponds to those regions where $\sigma^2 > \kappa^2(r)$ and is separated into the inner region of trapped oscillations, from $r = r_{\rm ms}$ to $r \approx (7/6)\, r_{\rm ms}$, and an outer region extending to $r \gg r_{\rm ms}$ (Kato et al., 1998). However, the lowest frequency modes in the continuum, which satisfy $\sigma < \kappa_{\rm max}$, are transmitted through the epicyclic barrier, and thus fluctuations in the inner disk may be transmitted to the outer disk for frequencies close to the maximum of the epicyclic one, $\sigma^2 \approx \kappa^2_{\rm max}$.

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