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Embedding diagrams of Bardeen geometry

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ABSTRACT
General relativity combined with a non-linear electrodynamics enables to find regular black hole solutions. The best known solution of this kind is described by the Bardeen spacetime with spacetime parameters giving gravitational mass m and magnetic charge g. For ratio g/m large enough, the Bardeen spacetime describes a no-horizon regular solution. Here we demonstrate properties of the Bardeen spacetimes by the embedding diagrams of the equatorial plane of the ordinary geometry, and the optical geometry enabling reflection of properties of test particle motion.

Keywords: Bardeen geometry – black hole – embedding diagram

1 INTRODUCTION

Black holes predicted by the general relativity contain a physical singularity with diverging Riemann tensor components. Regular black hole solutions of the Einstein gravity have been found that eliminate the physical singularity from the spacetimes having an event horizon, but these are not vacuum solutions of the Einstein equations, but contain necessarily a properly chosen additional field, or modified gravity.

The well known regular spherically symmetric black hole solution containing a magnetic charge as a source has been proposed by Bardeen (1968). The magnetic charge is related to a non-linear electrodynamics (Ayón-Beato and García, 2000). The solution is characterized by the mass parameter m and the charge parameter g. Their geodesic structure is governed by the dimensionless ratio g/m. For properly chosen charge parameter g/m, the Bardeen solution allows for existence of fully regular spacetime, without an event horizon. We call it Bardeen "no-horizon" spacetime.

A detailed discussion of the geodesic structure of the regular Bardeen black hole and no-horizon spacetimes and its implication to optical phenomena were presented in Stuchlík and Schee (2014a). It has been shown that the geodesic structure of the regular Bardeen black holes outside the horizon is similar to those of the Schwarzschild or Reissner–Nordström (RN) black hole spacetimes, but under the inner horizon, no circular geodesics can exist. The geodesic structure of the Bardeen no-horizon spacetimes is similar to those of the naked singularity spacetimes of the RN type, or the Kehagias–Sfetsos (KS) type (Kehagias and Sfetsos, 2009; Stuchlík and Schee, 2014b; Stuchlík et al., 2014) that is related to the solution of the modified Hořava quantum gravity (Hořava, 2009a,b). In all of these no-horizon and naked singularity spacetimes, an "antigravity" sphere exists consisting of static particles located at stable equilibrium points at a given "static" radius that can be surrounded by a Keplerian disc Stuchlík and Schee (2014b).

The basic properties of the Bardeen black hole and no-horizon spacetimes can be reflected by the embedding diagrams that illustrate in a proper way the curvature of the spacelike (constant time) surfaces and give for the ordinary space geometry an overall insight into its nature – (see e.g. Kristiansson et al. (1998); Stuchlík and Hledík (1999, 2002)). In the case of the optical geometry, the embeddings can give an illustration of some hidden properties of the geodesic structure of the spacetime (Stuchlík et al., 2000). Here we present the embeddings for both the Bardeen black hole and no-horizon spacetimes.

2 BARDEEN SPACETIMES

The spherically symmetric geometry of the regular Bardeen black-hole or no-horizon spacetimes is characterized in the standard spherical coordinates and the geometric units (c=G=1) by the line element

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{1}{f(r)}\,\mathrm{d}r^2 + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big)\,, \tag{1}$$

where the "lapse" $f(r)$ function depends only on the radial coordinate, the gravitational mass parameter m and the charge parameter g. The Bardeen spacetimes are constructed to be regular everywhere, i.e. the components of the Riemann tensor, and the Ricci scalar are finite at all $r \geq 0$ (Ayón-Beato and García, 1999).

The lapse function $f(r)$ reads

$$f(r) = 1 - \frac{2mr^2}{\big(g^2 + r^2\big)^{3/2}}\,. \tag{2}$$

The event horizons of the Bardeen black hole spacetimes, determined by the condition $f(r) = 0$, are given by

$$g^6 + \big(3g^2 - 4m^2\big)r^4 + 3g^4r^2 + r^6 = 0\,. \tag{3}$$

The critical value of the dimensionless parameter g/m separating the black-hole and the "no-horizon" Bardeen spacetimes reads

$$(g/m)_{\mathrm{NoH/B}} = 0.7698\,. \tag{4}$$

In the "no horizon" Bardeen spacetimes the metric is regular at all radii $r \geq 0$. We assume $r = 0$ to be the site of the self-gravitating charged source of the spacetime.

The optical geometry of the Bardeen spacetimes is given by the line element (Kristiansson et al., 1998)

$$\mathrm{d}s^2_{\mathrm{opt}} = -\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)^2} + \frac{1}{f(r)}r^2\,\mathrm{d}\theta^2 + \frac{r^2}{f(r)}\sin^2\theta\,\mathrm{d}\varphi^2\,. \tag{5}$$

3 THE EMBEDDING PROCEDURE

We make the embedding of the equatorial plane, $\theta = \pi/2$, of the $t = \text{const}$ spacelike sections of the spacetime and its optical geometry. For the ordinary, simply projected space, the 2D equatorial plane can be cast in the form

$$\mathrm{d}l_{\mathrm{ord}}^2 = \frac{\mathrm{d}r^2}{f(r)} + r^2\,\mathrm{d}\varphi^2\,, \tag{6}$$

while for the optical geometry we find

$$\mathrm{d}l_{\mathrm{opt}}^2 = \frac{\mathrm{d}r^2}{f(r)^2} + \frac{r^2}{f(r)}\,\mathrm{d}\varphi^2\,. \tag{7}$$

The plane has to be embedded into the 3D flat space with line element

$$\mathrm{d}l_{\mathrm{3D}}^2 = \mathrm{d}R^2 + R^2\,\mathrm{d}\phi^2 + \mathrm{d}\,z^2\,. \tag{8}$$

The 3D flat space is expressed in the standard cylindrical coordinates R, z, ϕ. The embedding is realized by the function $Z = Z(R)$ that implies the line element of the 2D embedding surface in the form

$$\mathrm{d}l_{\mathrm{2D}}^2 = \mathrm{d}R^2 + R^2\,\mathrm{d}\phi^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}R}\right)^2\,\mathrm{d}R^2\,. \tag{9}$$

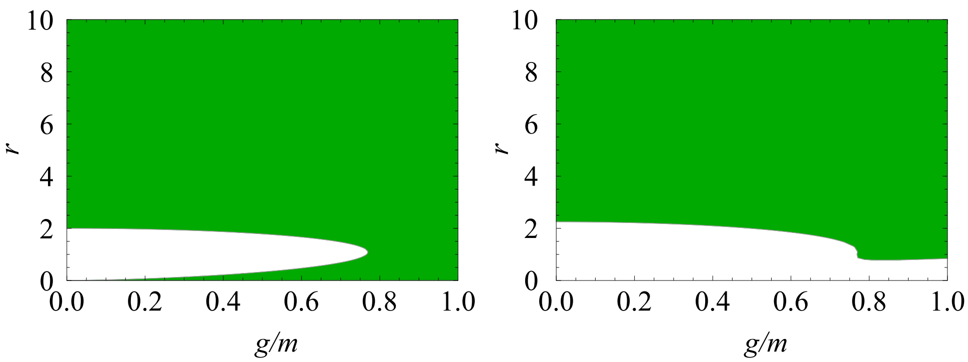

Figure 1. Embeddability limits of equatorial plane of directly projected (*left*) and optical (*right*) Bardeen geometry. *Green colour* indicates area that can be embedded.

4 EMBEDDING DIAGRAMS OF DIRECTLY PROJECTED GEOMETRY

In this case we can make the trivial identification

$$\phi = \varphi\,, \qquad R = r \tag{10}$$

that implies the relation

$$\left(1+\left(\frac{\mathrm{d}z}{\mathrm{d}r^2}\right)^2\right)=\frac{1}{f(r)}\,. \tag{11}$$

The embedding formula then takes a simple form

$$z=\int\sqrt{\tfrac{1}{f(r)}-1}\,\mathrm{d}r\,. \tag{12}$$

The embeddability conditions read

$$\frac{1}{f(r)}-1\geq 0\,,\qquad f(r)\geq 0\,. \tag{13}$$

Clearly, the region between the horizons is not embeddable. The regions are given in dependence on the spacetime parameter g/m in Fig. 1. The embedding diagrams are for representative values of the parameter g/m given in Fig. 2.

5 EMBEDDING DIAGRAMS OF THE OPTICAL GEOMETRY

In the case of the optical geometry, the identification of the radial coordinate is not trivial, we have to define

$$\phi=\varphi\,,\qquad R=\frac{r}{f^{1/2}(r)}\,. \tag{14}$$

Such an identification implies the relation

$$\left(\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)^2+\left(\frac{\mathrm{d}z}{\mathrm{d}r^2}\right)^2\right)=\frac{1}{f^{1/2}(r)} \tag{15}$$

and the embedding formula takes the form

$$z=\int\sqrt{\tfrac{1}{f^2(r)}-\left(\tfrac{\mathrm{d}R}{\mathrm{d}r}\right)^2}\,\mathrm{d}r\,. \tag{16}$$

The embeddability condition of the optical space reads

$$\frac{1}{f^2(r)}-\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)^2\geq 0\,. \tag{17}$$

The limits on the embeddability are given in Fig. 1, while the typical embedding diagrams of the optical space are illustrated in Fig. 3.

Recall that the turning points of the embeddings of the optical space reflect an important information on the geodesic structure of the spacetime, namely they represent loci of the photon circular orbits (Stuchlík et al., 2000).

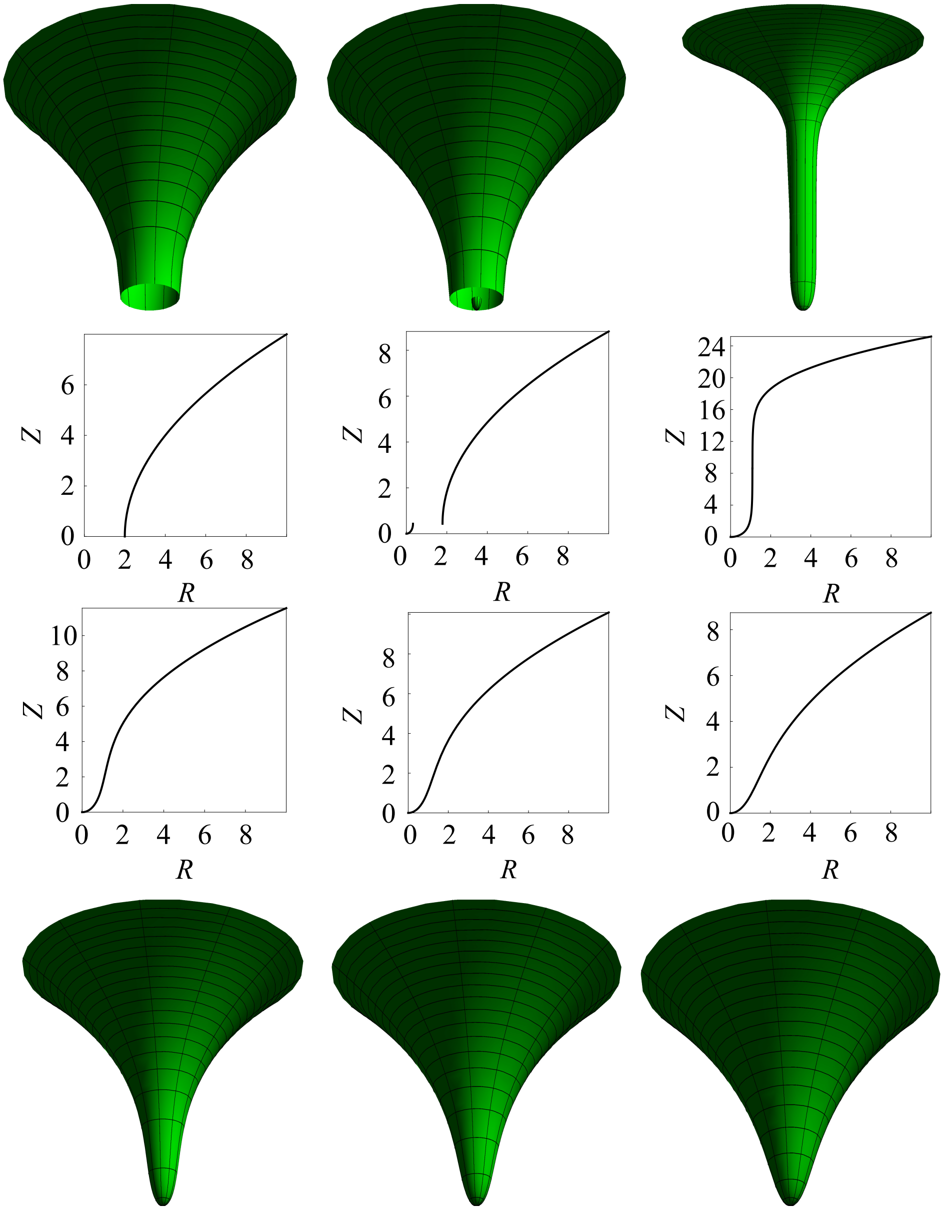

Figure 2. Embedding diagrams of directly projected geometry for different values of g/m. The integration in (12) ends slightly before horizons, because on the horizons there is $\mathrm{d}Z/\mathrm{d}r = 0$. Top part corresponds to $g/m = 0$ (Schwarzschild geometry), $g/m = 0.5$ (Bardeen black hole) and $g/m = 0.7698$ (extreme Bardeen black hole). Bottom part corresponds to $q/m = 0.8$ (there are two turning points in optical geometry, see Fig. 3), $g/m = 0.858665$ (there is one turning point in optical geometry) and $g/m = 1.0$ (there are no turning points in optical geometry).

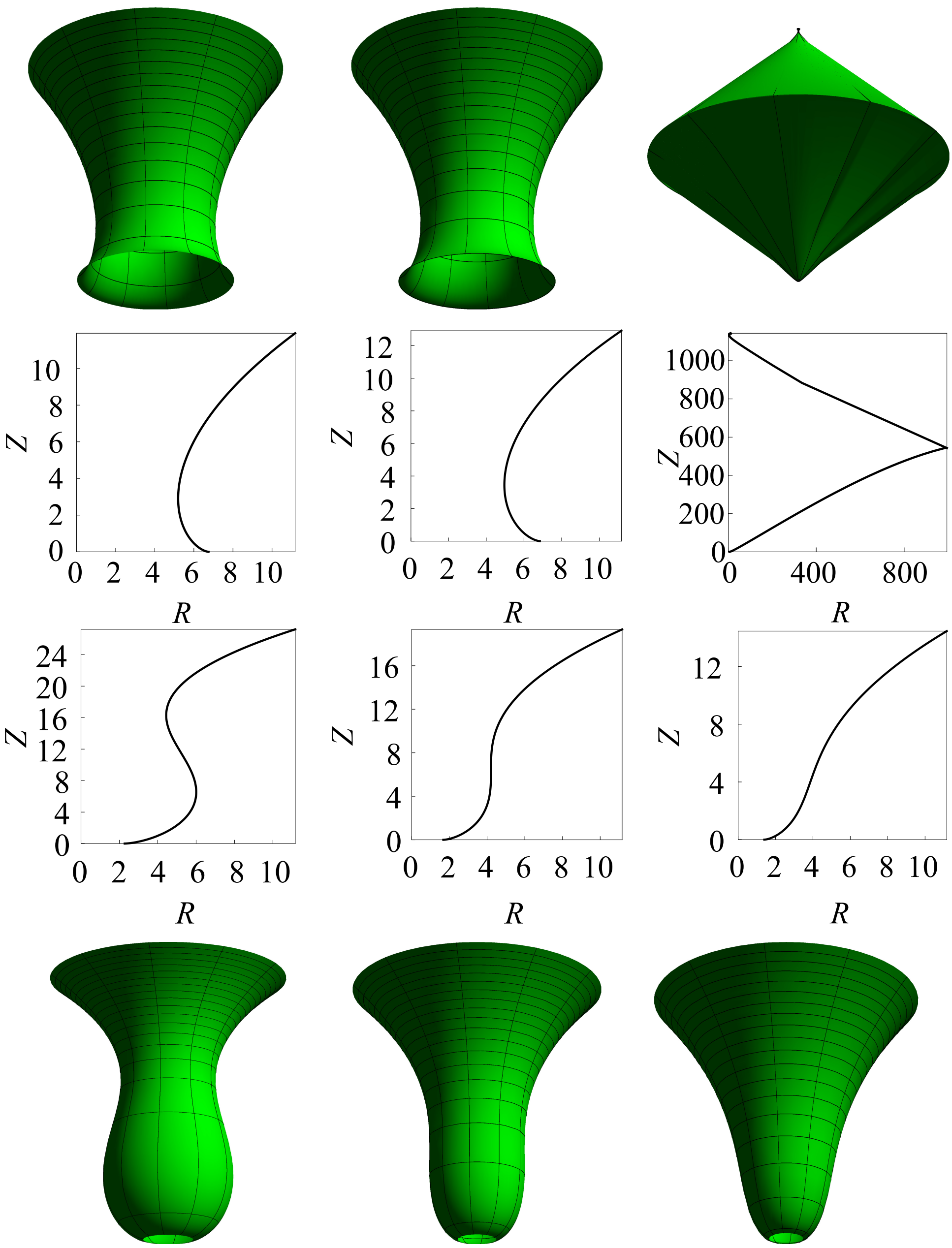

Figure 3. Embedding diagrams of optical geometry for different values of g/m. The integration in (16) ends slightly before embeddability limits for the same reason as in the normal case. The g/m values are exactly the same as on Figure 3.

6 CONCLUSIONS

We have constructed the embedding diagrams of the equatorial plane of the spherically symmetric regular Bardeen black-hole and no-horizon spacetimes for both the ordinary projected space, and the optical space. We have found the limits of embeddability of these spaces. The embeddability limits appear to be more extended in the vicinity of the coordinate origin $r = 0$ while compared to those related to the embeddings of the Kehagios–Sfetsos spacetimes that are spherically symmetric solutions of modified Hořava quantum gravity (Goluchová et al., 2014). This is rather surprising result, as the Kehagias–Sfetsos spacetimes are singular at $r = 0$, while the Bardeen spacetimes are regular there. The reason is related to different character of the "antigravity" region occurring near the origin of both Kehagias–Sfetsos and Bardeen spacetimes (Vieira et al., 2014; Stuchlík and Schee, 2014b,a). The gravitation repulsion in the Bardeen spacetimes occurring near the coordinate origin is of the de Sitter character, while in the case of the Kehagias–Sfetsos spacetimes, it is much weaker, being of a quintessential character Stuchlík and Schee (2014b,a).

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