

# Modified Newtonian potentials for particles and fluids in permanent rotation around black holes

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## ABSTRACT

Modified Newtonian potentials have been proposed for the description of relativistic effects acting on particles and fluids in permanent orbital motion around black holes. Here we further discuss spherically symmetric potentials like the one proposed by Artemova, Björnson & Novikov (1996, *Astrophysical Journal*, 461, 565), and we illustrate their virtues by studying the acceleration along circular trajectories. We compare the results with exact expressions in the spacetime of a rotating (Kerr) black hole.

**Keywords:** Accretion: accretion discs – black-hole physics

## 1 INTRODUCTION

The motion of material around black holes, both particles and fluids, is of particular importance for the present-day models of some astronomical objects, such as galactic X-ray sources and active galactic nuclei. In these systems, matter may be found rather close to the black-hole horizon, at a few gravitational radii ( $r_g = 2GM/c^2$ ; where  $M$  is the mass of the central black hole), and the effects of general relativity on the motion must be taken into account. (We will here set  $c = G = 1$ ; in addition, we will measure lengths in units of  $M$ , so that  $r_g = 2$  hereafter.) The relevant framework for discussing such fluids is then the Kerr spacetime of a rotating black hole (we consider here only test particles and fluids around the black hole, as is often done for astrophysical situations; however, for exact solutions of the Einstein equations with rotating bodies, see e.g. Islam (1985), and for more astrophysically realistic numerical solutions with self-gravitating material, see Lanza (1992); Nishida and Eriguchi (1994)). The relativistic effects for this matter can be ascribed mainly to two characteristic properties of motion around black holes: (i) presence of the marginally stable orbit ( $r = r_{ms}$ ) and the marginally bound orbit ( $r = r_{mb}$ ) which determine the regions of stable and energetically bound motion (their location determines also the inner edge of the toroidal fluid configurations; the exact location of  $r_{mb}$  and  $r_{ms}$  can be found by studying the effective potential; see Bardeen et al. (1972)); (ii) the frame

dragging of non-equatorial orbits (Lense–Thirring precession, often called the Wilkins (1972) effect in the case of motion close to a Kerr black hole). In order to incorporate these effects within the Newtonian framework (which is of course technically easier than a fully relativistic self-consistent approach), numerous authors have adopted the original idea of Paczyński and Wiita (1980) and employed modifications of the Newtonian potential (Nowak and Wagoner, 1991; Artemova et al., 1996; Crispino et al., 2011).<sup>1</sup>

In this Note we want to discuss simple (spherically symmetric) potentials appropriate for the description of matter in purely rotational motion, neglecting frame-dragging effects. This simplifies our discussion; see Semerák and Karas (1999) for detailed discussion and references concerning how to modify the Newtonian potential for including the effects of dragging. In previous studies, the main concern was about how to reproduce correctly the marginally bound and marginally stable orbits, since the properties of fluid tori are sensitive to the location of *both* of these orbits (cf. Muchotrzeb and Paczyński, 1982; Abramowicz et al., 1988; Kato et al., 1988; Chakrabarti, 1990). See also Tejeda and Rosswog (2013) and Barausse and Lehner (2013) for a recent discussion and new developments.

The Paczyński–Wiita potential,  $\Phi_{\text{PW}} = -1/(r - r_{\text{g}})$ , reproduces the correct location of  $r_{\text{mb}}$  and  $r_{\text{ms}}$  for a non-rotating black hole. Another form of the modified potential around a non-rotating black hole was used by Nowak and Wagoner (1991) to study relativistic wave-modes in accretion discs:  $\Phi_{\text{NW}} = -r^{-1} + 3r^{-2} - 12r^{-3}$  reproduces  $r_{\text{ms}}$  and the epicyclic frequency of radial oscillations  $\kappa$ . These two potentials  $\Phi_{\text{PW}}$  and  $\Phi_{\text{NW}}$  are not however applicable in the case of a rotating black hole. This situation has been treated by several authors, most recently and successfully by Artemova et al. (1996). Here we will further discuss the form of the potential which appears most convenient for modelling tori around rotating black holes. Note that tori rotate with non-Keplerian orbital velocity and they may extend well out of equatorial plane (Frank et al., 1992). One thus needs to consider also accelerated motion, though still in permanent rotation about the common axis of the black hole.

## 2 MODIFIED NEWTONIAN POTENTIAL FOR ROTATING BLACK HOLES

### 2.1 Motivation

The need for a practical and accurate modified potential leads to constraining its form according to the following conditions:

- (i) The modified potential should be a *simple* scalar function of the spherical radius  $r$ ;
- (ii) The potential should reduce to  $\Phi_{\text{PW}}$  in the limit of zero rotation (black-hole angular momentum parameter  $a = 0$ );
- (iii) The locations of  $r_{\text{ms}}$  and  $r_{\text{mb}}$  should be correctly reproduced *both* for the non-rotating case ( $a = 0$ ,  $r_{\text{ms}} = 6$ ,  $r_{\text{mb}} = 4$ ) and for the extreme rotating case ( $a = 1$ ,  $r_{\text{mb}} = r_{\text{ms}} = 1$ ).

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<sup>1</sup> A different approach was adopted in Keres (1967); Israel (1970); de Felice (1980), where some properties of the Kerr metric are described in terms of an axially symmetric (non-spherical) potential which reflects the asymptotic properties of test particle motion.

These requirements are satisfied by the function

$$\Phi = -\frac{1}{(r - r_+)^{\bar{\beta}} r^{1-\bar{\beta}}}, \quad (1)$$

where  $r_+ = 1 + \sqrt{1 - a^2}$  is the black hole outer horizon, and the parameter  $\bar{\beta}(a)$  is a free function. The choice for this is constrained by imposing that the values of  $r_{\text{mb}}$  and  $r_{\text{ms}}$  should be exact in the Schwarzschild case and in the extreme Kerr case. We then adopt the simplest linear form:  $\bar{\beta} = 1 - a$ . Although, for  $\bar{\beta} = 0$ , Eq. (1) reduces to the Newtonian potential which does not have marginally bound and marginally stable orbits, the correct location of  $r_{\text{mb}} = r_{\text{ms}} = 1$  is nevertheless obtained in the limit of  $a \rightarrow 1$  (the extreme Kerr case).

One can verify that the properties of the potential (1) are almost identical with those of the potential  $\Phi_{\text{ABN}}$  of Frank et al. (1992) which corresponds to their Eq. (13) for the force:

$$F_5 = -\frac{1}{r^{2-\beta} (r - r_+)^{\beta}}, \quad (2)$$

with  $\beta = (r_{\text{ms}}/r_+) - 1$ . Expression (2) follows from the following conditions:

- (i) The free-fall acceleration has a similar form to that for a Schwarzschild black hole;
- (ii) The free-fall acceleration diverges to infinity near  $r = r_+$ .
- (iii) The marginally stable orbit is reproduced *exactly for all* values of  $a$  ( $0 \leq a \leq 1$ ).

Although the position of the important orbit  $r = r_{\text{mb}}$  is not mentioned in the derivation of  $F_5$ , one can verify that the correct sequence is maintained for all  $a$ :  $r_+ \leq r_{\text{mb}} \leq r_{\text{ms}} \leq 6$  (indeed, the accuracy is very good as we will see in the next paragraph). The potential corresponding to  $F_5$  is

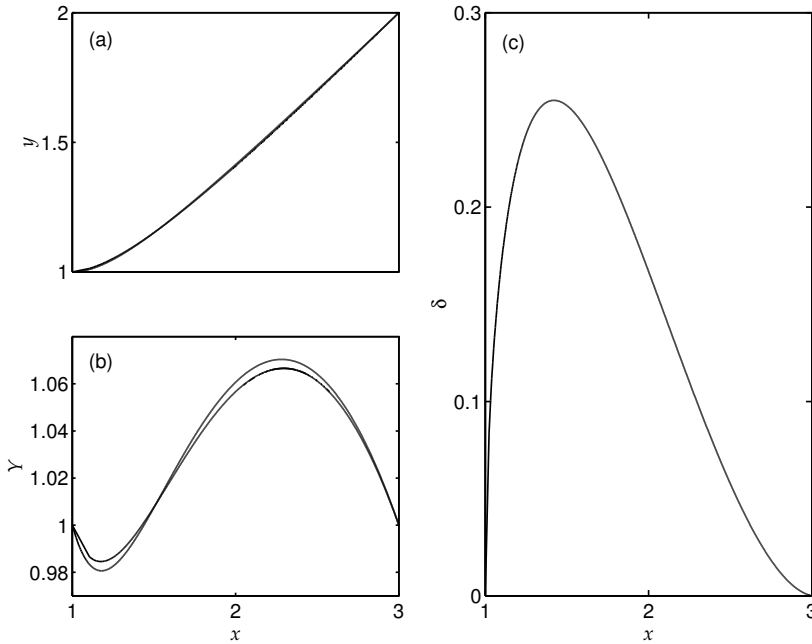
$$\Phi_{\text{ABN}} = \frac{1}{(1 - \beta)r_+} \left(1 - \frac{r_+}{r}\right)^{1-\beta} - \Phi_{\infty}, \quad (3)$$

with  $\Phi_{\infty} = (1 - \beta)^{-1}r_+^{-1}$ .

## 2.2 Acceleration along circular orbits

We will now argue that the results for purely rotational motion of fluids in potentials (1) and (3) should be extremely similar and close to the exact relativistic treatment in the Kerr metric. This conjecture can be illustrated in two steps: first we will see that  $r_{\text{ms}}$  and  $r_{\text{mb}}$  are well reproduced (for both  $\Phi$  and  $\Phi_{\text{ABN}}$ ), and then we will study acceleration along non-Keplerian circular orbits (relevant for modelling tori).

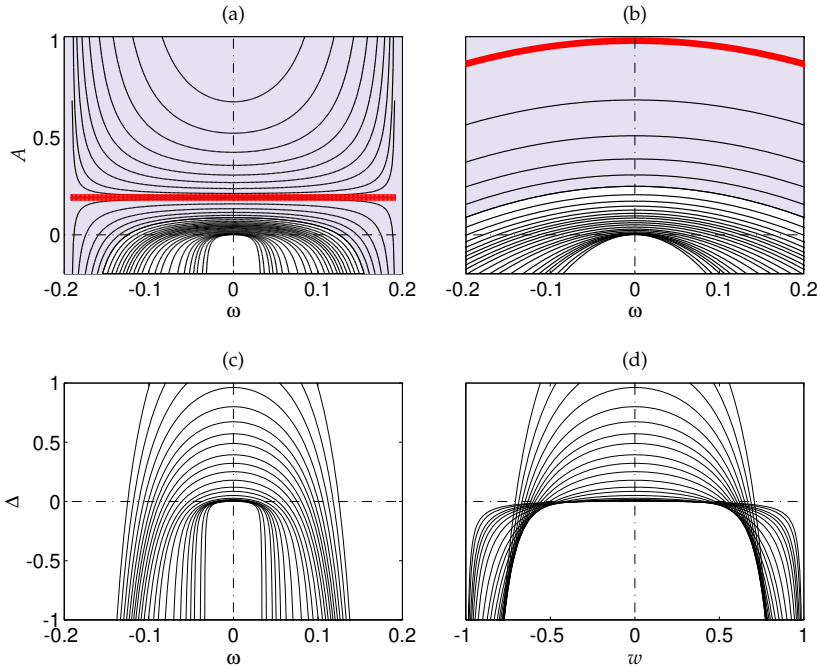
We now illustrate the differences in the marginally bound radius as a function of the marginally stable radius. Figure 1 compares our modified Newtonian ratios  $y = r_{\text{mb}}/r_+$  and  $x = r_{\text{ms}}/r_+$  (evaluated by using Eq. 1) with the corresponding values of the Boyer–Lindquist radial coordinate in the Kerr metric. In both cases  $1 \leq x(a) \leq 3$  and  $1 \leq y(a) \leq 2$  when the angular-momentum parameter varies in the range  $1 \geq a \geq 0$ . Figure 1a shows that the two curves of  $y(x)$  (i.e. the modified Newtonian and Kerr cases) are practically



**Figure 1.** Radii of important orbits in the modified Newtonian potential  $\Phi$  compared with the Kerr case: (a) the radius of the marginally bound orbit  $y$  plotted as a function of the radius of the marginally stable orbit  $x$ , both measured in units of the black hole outer horizon radius  $r_+$ ; (b) the normalized marginally bound radius  $Y(x) = fy$ ; (c) the relative difference  $\delta(x)$  between the modified Newtonian and Kerr cases (see text for definitions).

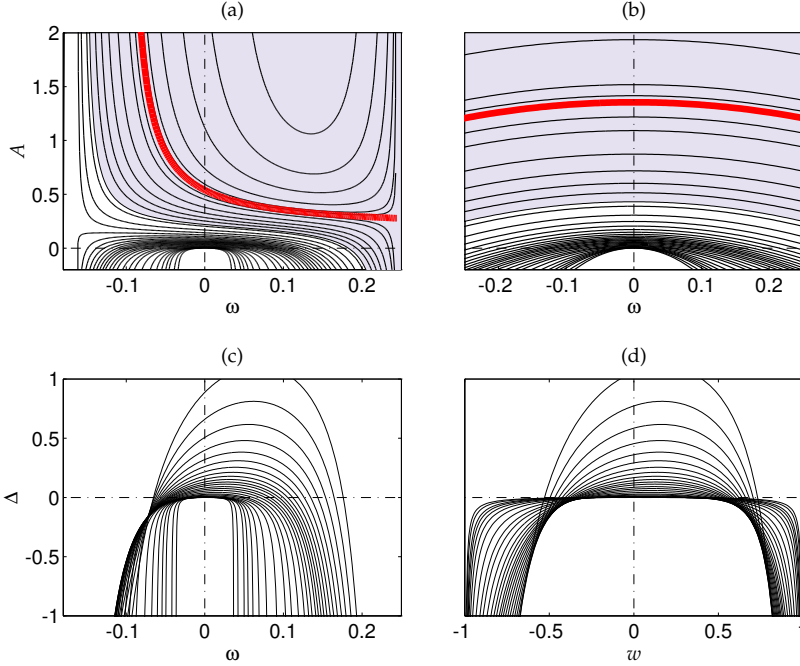
indistinguishable. In order to amplify the tiny difference, we introduce  $Y = f \cdot y$  where the normalization factor is given by  $f = 1 - (x - 1)/4$ . The curves of  $Y(x)$  are plotted in Fig. 1b. We complement these graphs by showing (Fig. 1c)  $\delta = \sqrt{[\delta x]^2 + [\delta y]^2}$  where  $\delta x(a)$ ,  $\delta y(a)$  are the differences in  $x$  and  $y$  between the modified Newtonian and Kerr cases. It can be seen that  $\delta \lesssim 0.25$ , which indicates an accuracy of  $x$  and  $y$  better than about 20%. The error is a maximum at  $x \approx 1.4$  and it goes sharply to zero for *both*  $x = 1$  ( $a = 1$ ) and  $x = 3$  ( $a = 0$ ). One can construct analogous graphs for  $\Phi_{\text{ABN}}$  but the results are very similar to those for  $\Phi$ . It is therefore a matter of taste which potential to choose for studying toroids in modified Newtonian potentials, but  $\Phi_{\text{ABN}}$  is perhaps more practical as it has already been used by other authors (Miwa et al., 1998).

The structure of relativistic tori is determined by the radial acceleration along circular trajectories, and of course by the pressure gradient which, however, depends on the equation of state. We will therefore now discuss the radial acceleration for different  $r = \text{const}$  and different angular velocity  $\omega$ , and again we will compare the case of the modified Newtonian potential with that of the Kerr metric (free circular orbits in the equatorial plane have  $\omega = 1/(r^{3/2} + a)$  and acceleration magnitude  $A = 0$ , but we do not restrict only to such



**Figure 2.** The acceleration magnitude  $A$  is plotted as a function of the angular velocity  $\omega$  along circular orbits with different radii: (a) the Schwarzschild case; (b) the modified Newtonian case  $\Phi_{\text{pw}}$  (the two plots are clearly different in the shaded area corresponding to  $r < r_{\text{mb}}$  but they are quite similar outside that region, i.e. in the bottom part of the plots); (c) and (d) show the relative difference  $\Delta$  between the two cases for radii in the range  $r_{\text{mb}} \leq r \leq 15 \rho$ . Acceptable accuracy of  $|\Delta| \lesssim 0.1$  corresponds to  $|w| \lesssim 0.5$  and  $r \gtrsim 1.9 r_{\text{mb}}$ .

cases). First, to explain how the graphs are constructed, we compare the acceleration for  $\Phi_{\text{pw}}$  and for the Schwarzschild metric in Fig. 2. Each curve gives the magnitude of the acceleration  $A$  along  $r = \text{const}$  orbits in the equatorial plane ( $\theta = 90^\circ$ ) of the Schwarzschild metric. (Only the radial component contributes to the acceleration in the equatorial plane; a general expression valid also outside of the equatorial plane in Kerr spacetime was given explicitly by Semerák (1994)). The radius progressively increases for the individual curves going from top to bottom of the plot. It has been widely discussed in the literature (Abramowicz and Prasanna, 1990) that  $A(\omega) = \text{const}$  at the photon orbit; this is indicated by a thick horizontal line in Fig. 2a. In fact, for applications to tori, we are mainly interested in orbits with radii greater than that of the marginally bound orbit, and therefore the whole portion of the graph corresponding to  $r < r_{\text{mb}}$  is covered by shading. (In this region the modified Newtonian potential approach is not accurate.) One can compare the shape of the curves in the Schwarzschild case to the modified Newtonian case of  $\Phi_{\text{pw}}$  in Fig. 2b. The relative difference  $\Delta$  between corresponding  $A$ 's from graphs 2a and 2b is plotted in the next two graphs, 2c–d, showing  $\Delta(\omega)$  and  $\Delta(w)$  ( $w$  denotes the speed in the local frame of



**Figure 3.** As in Fig. 2 but for the Kerr  $a = 0.5$  case (a), and for the equivalent  $\Phi_{\text{ABN}}$  case (b). Quite naturally,  $|\Delta|$  is on average large for the orbits with small radius. Comparing with analogous graphs for  $\Phi_{\text{PW}}$ , potential  $\Phi_{\text{ABN}}$  diminishes  $|\Delta|$  to smaller values, and is thus more accurate when  $a$  is nonzero. The graph here is not symmetrical about  $\omega = 0$  (due to frame-dragging in the Kerr metric); accuracy is maintained to higher  $|w|$  for corotating motion. Here, an acceptable accuracy of  $|\Delta| \lesssim 0.1$  corresponds to  $|w| \lesssim 0.5$  and  $r \gtrsim 1.6 r_{\text{mb}}$ .

a non-rotating observer, which corresponds to angular velocity  $\omega$ ;  $-1 < w < 1$ ;  $r > r_{\text{mb}}$ ). Here, the dimensionless quantity  $\Delta$  is defined as

$$\Delta = \frac{A_{\text{Exact}}(\omega) - A_{\text{Modified Newtonian}}(\omega)}{A_{\text{Exact}}(\omega = 0)} \quad (4)$$

which is to be evaluated for fixed  $r$ ,  $\theta$  and  $a$ . The outermost curve in Fig. 2c (with the largest magnitudes of  $\Delta$ ) corresponds to  $r = r_{\text{mb}}$ , while the innermost one (passing close to  $\omega = 0$ ,  $\Delta = 0$ ) corresponds to  $r = 15 r_{+}$ .

Figure 3 is constructed in the same way as Fig. 2, but now it compares the Kerr  $a = 0.5$  case with the equivalent modified Newtonian  $\Phi_{\text{ABN}}$  case. By inspecting graphs with different  $a$  we checked that the accuracy of the modified Newtonian potentials  $\Phi_{\text{PW}}$  and  $\Phi_{\text{ABN}}$  (as measured by  $\Delta$ ) is comparable in the non-rotating case but  $\Phi_{\text{ABN}}$  is better as soon as  $a$  is non-negligible. A similar conclusion can be drawn for  $\Phi$  from Eq. (1), and also for circular orbits outside the equatorial plane. Analogous plots to those in Figs. 2–3 have been constructed with other sets of parameters. We find that acceptable accuracy of about 10% in terms of  $\Delta$  is guaranteed whenever  $r \gtrsim 1.5 r_{\text{mb}}$ .

### 3 CONCLUSIONS

We have systematically checked and briefly illustrated that general relativistic effects on purely circular orbits can be imitated in a modified Newtonian potential. We have verified the accuracy of such models for the potentials  $\Phi$  (Eq. 1) and  $\Phi_{\text{ABN}}$  (Eq. 3), finding that these two are comparable and that both offer higher accuracy than the usual Paczyński and Wiita (1980) potential when the angular-momentum parameter  $a$  is nonzero. By using our criterion concerning the relative accuracy of the acceleration along circular orbits,  $|\Delta| \lesssim 0.1$ , we see that one can use the potential  $\Phi_{\text{ABN}}$  of Artemova et al. (1996) satisfactorily for modelling tori in permanent orbital motion around a rotating black hole. The error increases very close to  $r_{\text{mb}}$ . The same conclusion holds for analogous potentials (such as the one proposed in this note, Eq. 1) which reproduce the important orbits and accelerations for motion around a rotating black hole with an acceptable accuracy.

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