

Eigenmodes of trapped horizontal oscillations in accretion disks

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ABSTRACT

We present eigenfrequencies and eigenfunctions of trapped acoustic-inertial oscillations of thin accretion disks for a Schwarzschild black hole and a rapidly rotating Newtonian star (a Maclaurin spheroid). The results are derived in the formalism of Nowak and Wagoner (1991) with the assumption that the oscillatory motion is parallel to the midplane of the disk. The first four radial modes for each of five azimuthal modes ($m = 0$ through $m = 4$) are presented. The frequencies and wavefunctions of the lowest modes may be accurately approximated by Airy's function.

Keywords: Relativistic stars: black holes – structure stability – oscillations – relativity – gravitation – accretion disks – hydrodynamics

1 TRAPPED MODES

Kato and Fukue (1980) showed that acoustic-inertial modes may be trapped in the inner parts of an accretion disk. This occurs when the (radial) epicyclic frequency κ has a maximum, as is the case in the Schwarzschild metric of general relativity (GR) considered by the authors. Okazaki et al. (1987); Kato (1989) and Nowak and Wagoner (1991, 1992) consider a model of a black hole accretion disk in hydrostatic equilibrium, and derive a dispersion relation for modes with $n = 0, 1, 2, 3, \dots$ nodes along the z -axis (the symmetry axis of the disk). The trapping occurs for oscillation frequencies below the maximum of the epicyclic frequency $\omega < \kappa_{\max}$. Here $\omega(r) = m\Omega(r) + \sigma$ is the frequency in the frame co-rotating with the fluid (at angular frequency Ω), m is the azimuthal mode number, σ is the eigenfrequency of the mode, and $\kappa^2 = (2\Omega/r)d(r^2\Omega)/dr$. The $n = 0$ modes will be trapped between the inner edge of the disk, close to the ISCO at $\kappa(r_{\text{ms}}) = 0$, and the lowest radius r satisfying $\omega(r) = \kappa(r)$, while for $n = 1$ trapping occurs close to the maximum of κ , between those two radii at which $\omega = \kappa$. Further discussion can be found in the textbook by Kato et al. (1998). In this contribution we only consider the $n = 0$ trapped modes.

Currently, the main interest in disk oscillations is related to the observed frequencies in the X-ray flux from black hole and neutron star systems (for a review see van der Klis M., 2000). For black hole disks the modes thought to be offering the most promising explanation (Wagoner et al., 2001) of the highest observed frequencies are the g -modes and

c -modes, investigated in full GR by Perez et al. (1997); Silbergleit et al. (2001), although a different explanation seems to be required for the observed 3:2 ratio of the highest frequencies in the microquasars (Abramowicz and Kluźniak, 2001; Kluźniak et al., 2004; Török et al., 2005). Thus, the modes investigated here are not prime candidates for a theoretical counterpart to the observed high frequency QPOs (quasi-periodic oscillations) in black hole systems. However, similar phenomena are observed in white dwarf systems (Woudt and Warner, 2002), and while their harmonic content may be explained by a resonance (Kluźniak et al., 2005), the origin of the observed frequencies remains obscure. For this reason we would like to discuss disk oscillations in a framework valid equally in a GR and non-GR context.

2 EQUATION OF MOTION AND THE BOUNDARY CONDITION

We will be closely following the approach of Nowak and Wagoner (1991) who describe perturbations with a Lagrangian displacement vector in cylindrical coordinates $(\xi_r^r, \xi_r^\phi, \xi_r^z) = (\xi^r, \xi^\phi, \xi^z) \exp[i(m\phi + \sigma t)]$ in the formalism of Friedman and Schutz (1978), and show

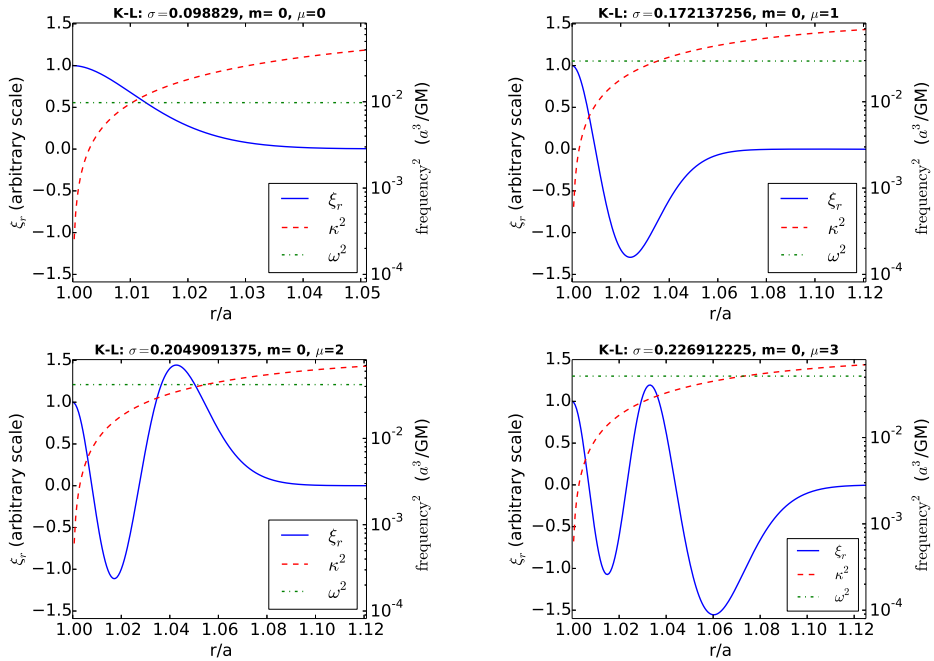


Figure 1. The fundamental and the first three radial overtones for $m = 0$ trapped horizontal oscillations of a thin ($H/a = 10^{-3}$) accretion disk for the potential of Eq. (4). Plotted are the wavefunction: solid (blue) line (arbitrary normalization, left scale); $\omega^2(r)/\Omega^2(r_{\text{ms}})$: dashed-dotted (green) line and $\kappa^2(r)/\Omega^2(r_{\text{ms}})$: dashed (red) line (logarithmic scale, right).

that in the WKB approximation the azimuthal component of the equation of perturbed motion for thin disks reduces to $\xi^\phi = 2i(\Omega/\omega)\xi^r$. In this contribution we assume horizontal motion, implying that $\xi_*^z \equiv 0$ and $\partial\xi_*^r/\partial z \equiv 0$. In terms of $\Psi(r) \equiv \sqrt{\gamma P r} \xi^r(r)$ the remaining component of the equation of motion then gives

$$\frac{d^2\Psi}{dr^2} + \frac{(\omega^2 - \kappa^2)}{c_s^2}\Psi = 0, \quad (1)$$

where $c_s^2 = \gamma P/\rho$ is the speed of sound squared; the boundary condition is that the Lagrangian perturbation of pressure vanishes at the unperturbed boundary, $\Delta P \equiv \gamma P \nabla \xi_* = 0$, which reduces to

$$\frac{1}{r} \frac{\partial}{\partial r}(r\xi_*^r) + \frac{1}{r} \frac{\partial}{\partial \phi}(\xi_*^\phi) = 0$$

assuming that $P \neq 0$ (Nowak and Wagoner, 1991). Neglecting derivatives of P this gives our final boundary condition at the inner edge, at $r = a$, which we will take to be at the marginally stable orbit (ISCO) at $a = r_{\text{ms}}$,

$$\frac{d\Psi}{dr} = -\frac{\Psi}{2r}(1 - 4m\Omega/\omega).$$

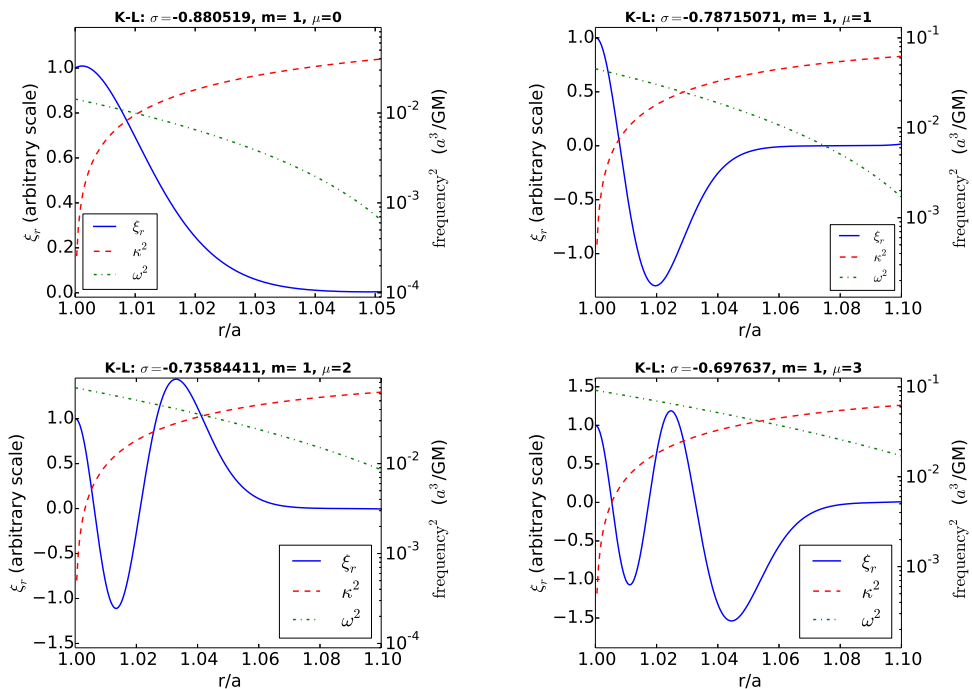


Figure 2. Same as Figure 1, but for $m = 1$.

In dimensionless form, with $r = a(1 + x)$, $\tilde{\omega}(x) = \omega(r)/\Omega(a)$, $\tilde{\kappa}(x) = \kappa(r)/\Omega(a)$, $\tilde{\sigma} = \sigma/\Omega(a)$, and $c_s = H\Omega(a)$, the perturbation (wave) equation takes the form

$$\frac{d^2\Psi}{dx^2} + \left(\frac{a}{H}\right)^2 (\tilde{\omega}^2 - \tilde{\kappa}^2) \Psi = 0, \quad (2)$$

with the boundary condition at $x = 0$

$$\frac{d\Psi}{dx} = -\frac{\Psi}{2}(1 - 4m/\tilde{\omega}). \quad (3)$$

In the last equation $\tilde{\omega} = \tilde{\sigma} + m$. Recall that in general $\tilde{\omega}(x) = \tilde{\sigma} + m\Omega(r)/\Omega(a)$.

In this contribution we are providing an atlas of eigenfrequencies and eigenfunctions for the fundamentals and the first three radial overtones of horizontal disk oscillations (labelled with the number of radial nodes, $\mu = 0, 1, 2, 3$) for $m = 0, 1, 2, 3, 4$.

3 MODELS OF A SCHWARZSCHILD BLACK HOLE

Bohdan Paczyński showed that it is possible to capture essential qualitative features of motion in the Schwarzschild metric in a Newtonian model with a simple pseudo-potential $\Phi(r) = -GM/(r - 2r_g)$ (Paczyński and Wiita, 1980), with $r_g = GM/c^2$. Nowak and

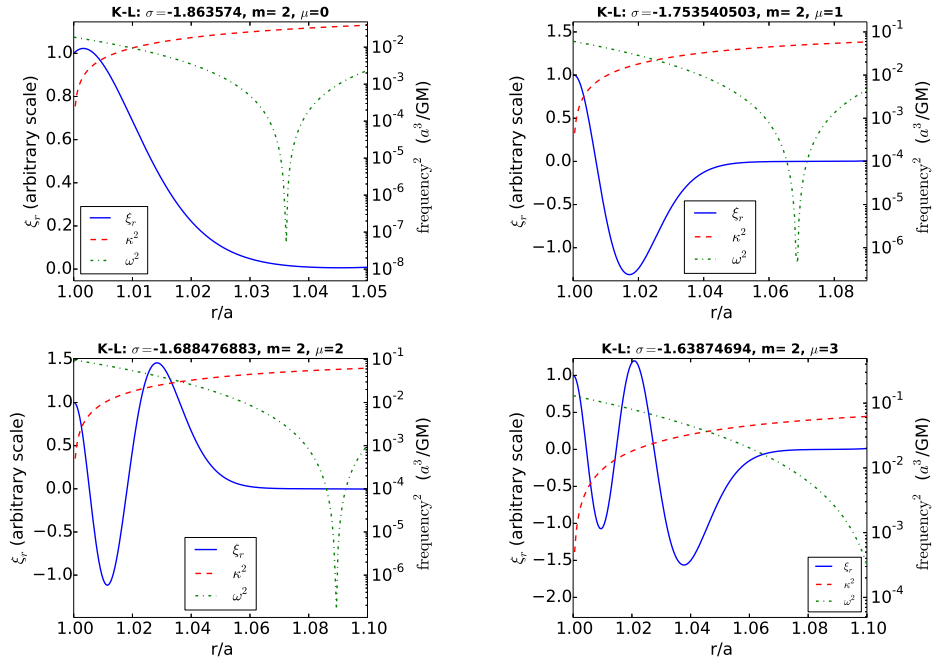


Figure 3. Same as Figure 1, but for $m = 2$.

Wagoner (1991) found the eigenfrequencies and eigenfunctions of Eq. (1) for the fundamental oscillations with $m = 0$, and $m = 2$, using values of $\kappa^2(r)$ following from their own pseudo-potential $\Phi(r) = -(GM/r)[1 - 6r_g/r + 12(r_g/r)^2]$.

Here, we model the Schwarzschild metric with a Newtonian pseudo-potential designed expressly to reproduce the Schwarzschild ratio of $\kappa^2(r)/\Omega^2(r) = 1 - 6r_g/r$:

$$\Phi_{\text{KL}}(r) = -(c^2/6) \exp(6r_g/r - 1). \quad (4)$$

As we are only interested in the inner parts of an accretion disk, we have dropped an additive constant. We have also renormalized the original form of the potential (Kluźniak and Lee, 2002) by a factor of $1/e$ to guarantee the correct value of $\Omega(r_{\text{ms}})$. The angular frequency of orbital motion follows from $\Omega^2(r) = r^{-1}\partial\Phi_{\text{KL}}/\partial r$ and, as for the other two potentials, the marginally stable orbit comes out to be at $r_{\text{ms}} = 6GM/c^2$. We have numerically solved the eigenvalue problem given by Eqs. (2) and (3), for $H/a = 10^{-3}$. The equations being linear in Ψ , we normalize the wavefunction to unity at the inner edge of the disk: $\Psi(r_{\text{ms}}) = 1$. Figure 1 presents the eigenfrequencies σ and the eigenfunctions $\Psi(r)$ for $m = 0$ and $\mu = 0, 1, 2, 3$, while Figs. 2, 3, 4 and 5 present the same quantities, as well as $\tilde{\omega}^2$, for $m = 1, 2, 3, 4$, respectively.

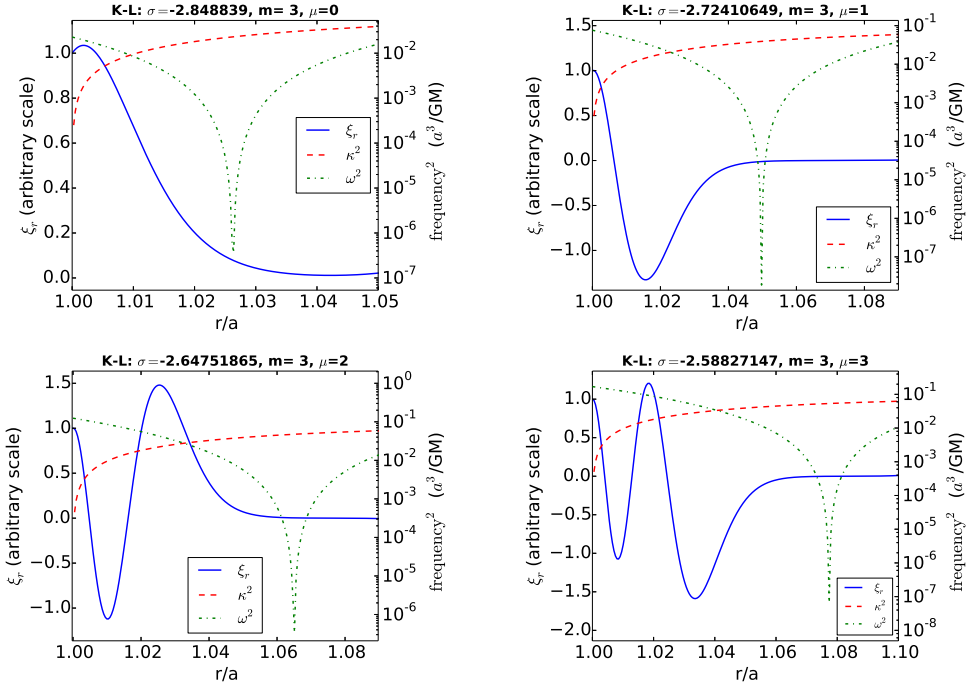


Figure 4. Same as Figure 1, but for $m = 3$.

4 ESSENTIALS OF ACOUSTIC-INERTIAL OSCILLATIONS

4.1 Wave equation

Dust orbiting an axially symmetric gravitating body in its equatorial plane ($z = 0$) would settle in stable circular orbits. The orbit of each dust particle being stable, it corresponds to “rest” (we are only concerned with radial motion in this section) at a fixed radial distance from the center of the body in the minimum of the effective potential, $V(r, z) = \Phi(r, z) + l^2/(2r^2)$, $l \equiv r^2\Omega(r)$ being the conserved angular momentum of a given particle, and Φ the gravitational potential of the body, both per unit mass. Consider small radial perturbations δr of motion of a dust disk (such as the rings of Saturn). Neglecting particle collisions, the perturbed dust would be executing radial harmonic (epicyclic) motion with respect of the stable orbits. The square of the frequency of this radial motion, $\kappa^2 = \partial^2 V/\partial^2 r$ corresponds to the strength of the restoring force per unit mass: $-\kappa^2\psi_*$ (if we denote the radial displacement $\delta r = \psi_*$). If the dust disk is replaced by a fluid, there will be an additional restoring force corresponding to pressure perturbations.

It is well known that sound waves in a homogeneous medium can be described by a harmonic function both in space and in time, with a constant and uniform amplitude if attenuation is neglected. Thus, the acoustic displacement of the fluid satisfies both a wave

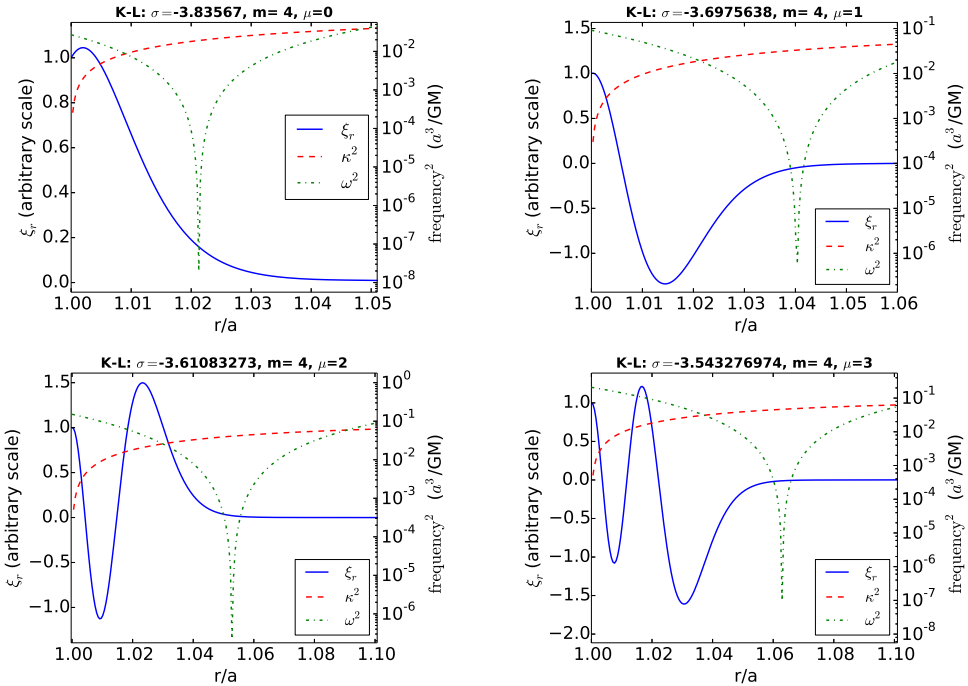


Figure 5. Same as Figure 1, but for $m = 4$.

equation

$$\frac{\partial^2 \psi_*}{\partial y^2} + k_s^2 \psi_* = 0, \quad (5)$$

and an oscillator equation

$$\frac{\partial^2 \psi_*}{\partial t^2} + \omega_s^2 \psi_* = 0, \quad (6)$$

corresponding to a restoring force $-\omega_s^2 \psi_*$. The frequency of the sound wave is related to the wave vector through the linear dispersion relation

$$k_s^2 = \omega_s^2 / c_s^2. \quad (7)$$

Clearly, taking into account in the oscillator equation both the ‘‘inertial’’ (epicyclic) and the acoustic restoring forces, and neglecting for the moment the difference between the cylindrical co-ordinate r and the Cartesian co-ordinate y , the acoustic-inertial displacement of the fluid can be described by a displacement $\psi_*(y, t) = \psi(y) \exp(i\omega t)$, with $\omega^2 =$

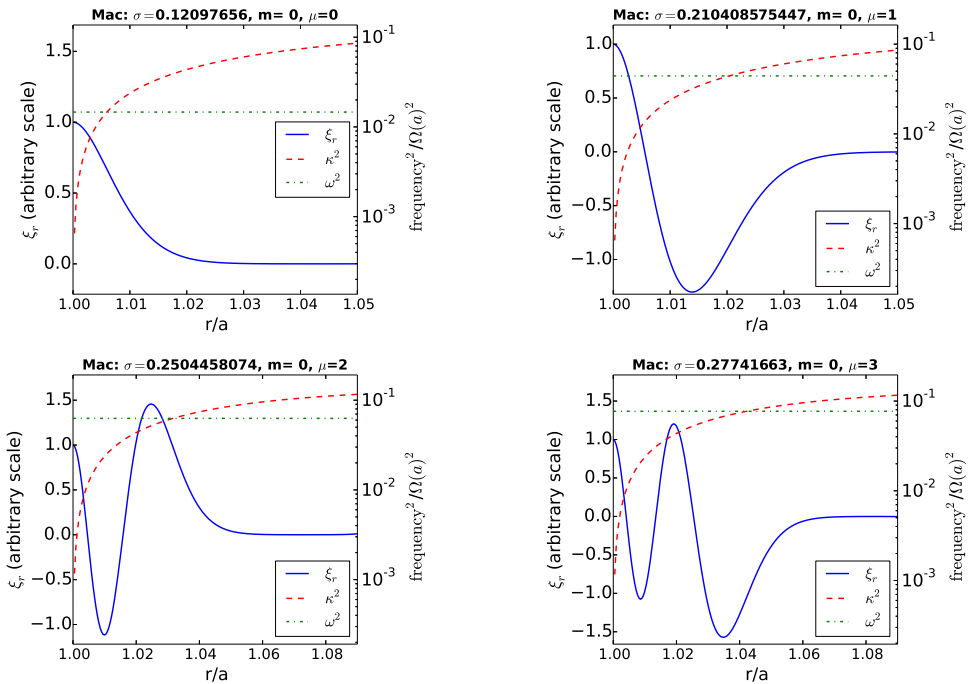


Figure 6. Same as Figure 1, but for the potential of a Maclaurin spheroid with ellipticity $e = 0.834583178$.

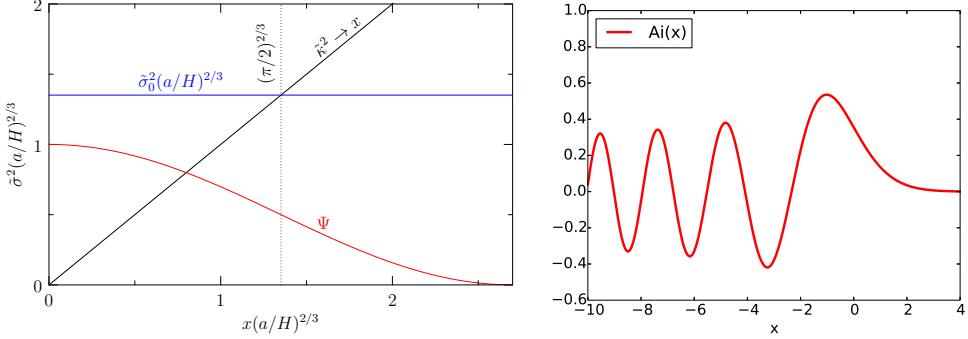


Figure 7. *Left Panel:* A simple approximation to the eigenfrequency $\tilde{\sigma}_0$ of the axisymmetric fundamental mode ($m = 0, \mu = 0$): a quarter wavelength fits in the inner region of the disk ($\tilde{\sigma}_0^2 \geq \tilde{\kappa}^2$) before the wave becomes evanescent. The vertical dotted line indicates the value of x at which $\tilde{\sigma}_0^2$ and x must intersect for this condition to be met: $\tilde{\sigma}_0^2 = \tilde{\kappa}^2(x_0) \approx x_0$, with $x_0 = (\pi/2)^{2/3} \cdot (H/a)^{2/3}$. The diagonal solid (black) line corresponds to the linear approximation $\tilde{\kappa}^2(x) = x + \mathcal{O}(x^2)$, which is valid for the potential of Eq. (4). See Equation (9) and Section 4.2 for details
Right Panel: A better approximation is obtained from the location of the extrema of Airy's function $\text{Ai}(X)$. Note that the shape of $\text{Ai}(X)$ closely resembles the shape of the numerically found $\Psi(x)$.

$\kappa^2 + \omega_s^2$ (or, in the form written down by Binney and Tremaine, 1987, $\omega^2 = \kappa^2 + k_s^2 c_s^2$). Substituting this new dispersion relation into Eq. (7), we see that Eq. (5) takes the form

$$\frac{d^2 \psi}{dy^2} + \frac{\omega^2 - \kappa^2}{c_s^2} \psi = 0. \quad (8)$$

Remarkably, this is the same equation that was rigorously derived by Nowak and Wagoner (1991), i.e. Eq. (1). In the remainder of this paper we will be discussing numerical solutions of its dimensionless version, Eq. (2), subject to the boundary condition Eq. (3), for a thin disk ($H/a = 0.001$) in two different models of the gravitating body, i.e. for two different epicyclic frequencies $\kappa(r)$.

4.2 Estimates of the eigenfrequencies

It is possible to understand the values of the eigenfrequencies σ and the shape of the wavefunctions in a simple model of Eq. (2). For axially symmetric modes, $m = 0$ and hence $\omega = \sigma$. The wave equation has oscillatory solutions for $\omega^2 > \kappa^2$, while the wave is evanescent for $\omega^2 < \kappa^2$. Thus the mode is trapped between $a = r_{\text{ms}}$ (i.e. $x = 0$) and $r = r_0$ such that $\sigma^2 = \kappa^2(r_0)$ (Fig. 7).

As $\sigma^2 \ll \kappa_{\text{max}}^2$ for the fundamental mode and $\kappa^2(r_{\text{ms}}) = 0$ we can model κ^2 with a linear approximation (Nowak and Wagoner, 1991), which for the potential of Eq. (2) has the simple form $\tilde{\kappa}^2 = x$. Thus, the wave becomes evanescent at $r_0/a - 1 = x_0 \approx \tilde{\sigma}^2$. We can take the boundary condition on the wave to correspond to that of a banner flapping in the wind, with a crest at the edge [of the disk ($x = 0$)] and a node close to x_0 . Perhaps a quarter wavelength of a sinusoid between $x = 0$ and $x = x_0$ is a fair approximation (Kato and Fukue, 1980).

With the above approximations, we have $ax_0 = \lambda/4$, and $\tilde{\sigma}^2 = x_0$. Now, $k = 2\pi/\lambda \approx \sigma/c_s$ so $\lambda/4 \approx \pi c_s/(2\sigma) = \pi H\Omega/(2\sigma)$. Recall that $\tilde{\sigma} = \sigma/\Omega(a)$. Finally, we obtain $\tilde{\sigma}^3 \approx \pi H/(2a)$, yielding

$$\tilde{\sigma} \approx \left(\frac{\pi H}{2a}\right)^{1/3} \approx 1.16 \left(\frac{H}{a}\right)^{1/3}. \quad (9)$$

For $H/a = 10^{-3}$ this yields $\tilde{\sigma} \approx 0.116$, while the numerically obtained value for the correct functional form of κ^2 is $\tilde{\sigma}_0 \approx 0.0988$. Thus, this crude estimate of the eigenfrequency is off by less than 20%. However, as we will see directly below, we have obtained the correct scaling of the eigenfrequency with the dimensionless thickness of the disk (Kato and Fukue, 1980).

A more accurate estimate of the eigenfrequency can be obtained by noting that in the linear approximation to κ^2 (which for the potential of Eq. (4) is simply $\tilde{\kappa}^2 = x$), Eq. (2) corresponds to Airy's equation (Nowak and Wagoner, 1991). Indeed, with the substitution $X = (x - \tilde{\sigma}^2)(a/H)^{2/3}$, Eq. (2) becomes $d^2\Psi/dX^2 = X\Psi$, with the Airy function as the solution: $\Psi(X) = \text{Ai}(X)$. In the exact waveforms of Fig. 1, one can recognize the shape of Airy's function, to a good accuracy. The (implicit) eigenvalues $\tilde{\sigma}$ can now be found directly

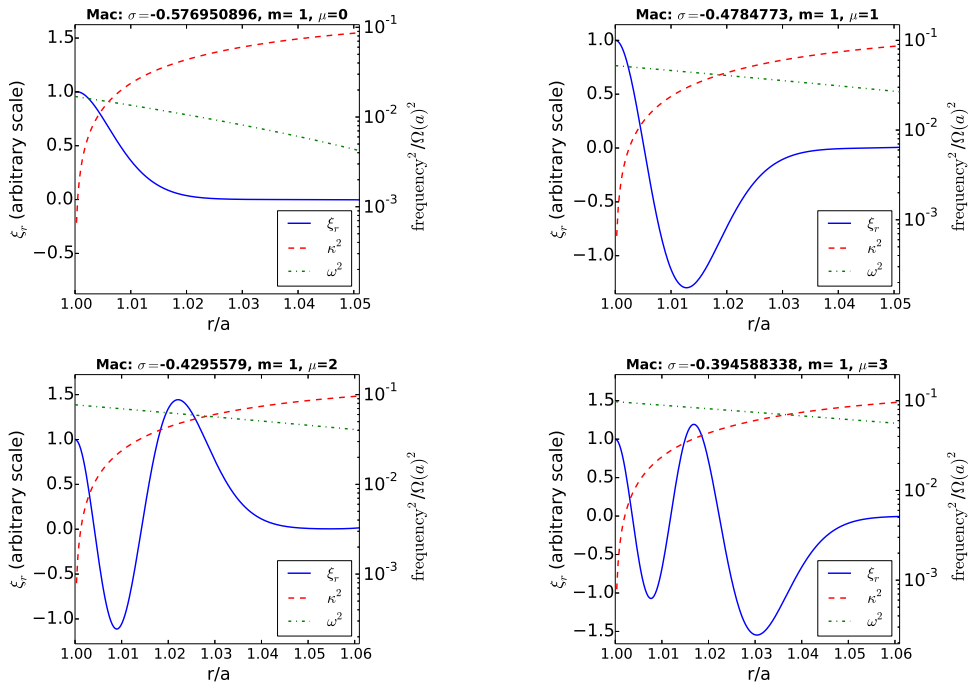


Figure 8. Same as Figure 6, but for $m = 1$.

from the boundary condition, Eq. (3), in the form

$$\frac{1}{\Psi} \frac{d\Psi}{dX} = \left(\frac{H}{a}\right)^{2/3} \frac{1}{2}(4m/\tilde{\omega} - 1).$$

Now, for $H \ll a$, the boundary condition (at $x = 0$) becomes $d \log \Psi/dX \ll 1$, i.e. it is approximately that X corresponds to one of those X_μ for which $\text{Ai}(X_\mu)$ has an extremum, $d\text{Ai}/dX|_{X_\mu} = 0$. Thus, $\tilde{\sigma}_\mu^2 \approx -(H/a)^{2/3} X_\mu$, $\mu = 0, 1, 2, 3, \dots$. We can compare these approximate eigenfrequencies (Table 1) with the numerically found eigenvalues for the correct form of κ^2 . For the fundamental the agreement is quite good, but the accuracy of the Airy approximation gradually degrades as σ_μ approaches the value κ_{\max} .

Table 1: Exact and approximate eigenvalues of Eqs. (2) and (3)

$m = 0, \quad H/a = 0.001$	$\mu = 0$	$\mu = 1$	$\mu = 2$	$\mu = 3$
$\tilde{\sigma}_\mu$ for κ^2 of Eq. (4)	0.0988...	0.172...	0.205...	0.227...
Airy approx.: $\sqrt{-0.01 X_\mu}$	0.101...	0.180...	0.229...	0.248...
Accuracy of approximation	2%	5%	7%	9%

We thank Mr. Luca Giussani for providing us with the values of Airy’s extrema.

Table 2: Eigenvalues of Eqs. (2) and (3)

$H/a = 0.001$	$\mu = 0$	$\mu = 1$	$\mu = 2$	$\mu = 3$
KL Eq. (4), $m = 0$. $\tilde{\sigma}_\mu =$	0.098829	0.172137	0.204909	0.226912
Maclaurin, $m = 0$. $\tilde{\sigma}_\mu =$	0.120977	0.210409	0.250446	0.277417
KL Eq. (4), $m = 1$. $\tilde{\sigma}_\mu =$	-0.880519	-0.787151	-0.735844	-0.697637
Maclaurin, $m = 1$. $\tilde{\sigma}_\mu =$	-0.576951	-0.478477	-0.429558	-0.394588
KL Eq. (4), $m = 2$. $\tilde{\sigma}_\mu =$	-1.86357	-1.753541	-1.688477	-1.638747
Maclaurin, $m = 2$. $\tilde{\sigma}_\mu =$	-1.275666	-1.168932	-1.112464	-1.07094
KL Eq. (4), $m = 3$. $\tilde{\sigma}_\mu =$	-2.848839	-2.724106	-2.647519	-2.588271
Maclaurin, $m = 3$. $\tilde{\sigma}_\mu =$	-1.974981	-1.860566	-1.797395	-1.750153
KL Eq. (4), $m = 4$. $\tilde{\sigma}_\mu =$	-3.83567	-3.697564	-3.610833	-3.543277
Maclaurin, $m = 4$. $\tilde{\sigma}_\mu =$	-2.674776	-2.553138	-2.483865	-2.431476

5 TRAPPED OSCILLATIONS IN AN ACCRETION DISK AROUND A MACLAURIN SPHEROID

In previous sections, following Nowak and Wagoner (1991) we were discussing the trapped acoustic-inertial oscillations of a pseudo-Newtonian model of an accretion disk around

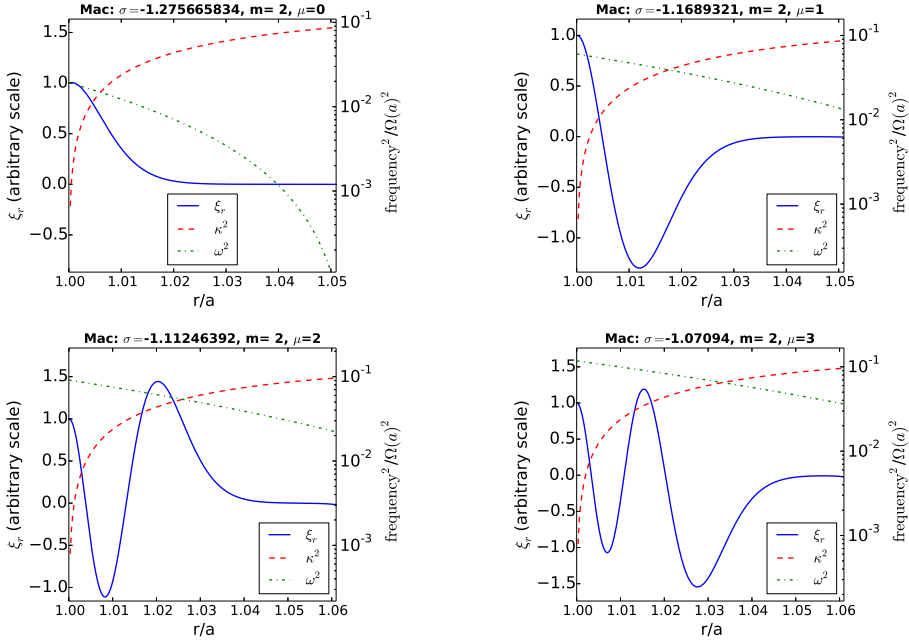


Figure 9. Same as Figure 6, but for $m = 2$.

a Schwarzschild black hole. Interestingly, the same trapping phenomenon occurs in strictly Newtonian gravity, for disks orbiting sufficiently oblate bodies. Kluźniak et al. (2001), and Zduńik and Gourgoulhon (2001) pointed out that oblateness of a gravitating body can destabilize orbits close to it, while Amsterdamski et al. (2002) showed that the marginally stable orbit exists in the Newtonian potential of classic Maclaurin spheroids for a sufficiently large ellipticity of the spheroid, i.e. a sufficiently large rotation rate of the spheroid. Kluźniak and Rosińska (2013) give explicit expressions for the angular velocity in circular orbits and for the corresponding epicyclic frequencies as a function of orbital radius and the ellipticity of the Maclaurin spheroid. Gondek-Rosińska et al. (2014) compare these analytic expressions with exact numerical solutions (in GR) of rapidly rotating quark stars, while Mishra and Vaidya (2014) give accretion disk solutions in the gravitational field of Maclaurin spheroids, which are reminiscent of the Shakura and Sunyaev (1973) black hole accretion disks.

Without further ado, we are presenting the eigenfrequencies and eigenvalues of trapped acoustic-inertial modes for an accretion disk around a Maclaurin spheroid of ellipticity $e = 0.834583178$. We are using the same equation and boundary conditions as before, Eqs. (2) and (3), with the functional form of $\kappa^2(r)$ and $\Omega^2(r)$ appropriate for the chosen Maclaurin spheroid. The only other change is that we need to reinterpret H : the condition of hydrostatic equilibrium is $c_s = h\Omega_\perp$, with h being the half-thickness of the disk, and Ω_\perp the vertical epicyclic frequency which we absorb into an effective half-thickness $H = h\Omega_\perp(a)/\Omega(a)$. The results are summarized in Figs. 6, 8, 9, 10, 11 for modes with $m = 0, 1, 2, 3, 4$, respectively. The frequencies are compared in Table (2) with those obtained in

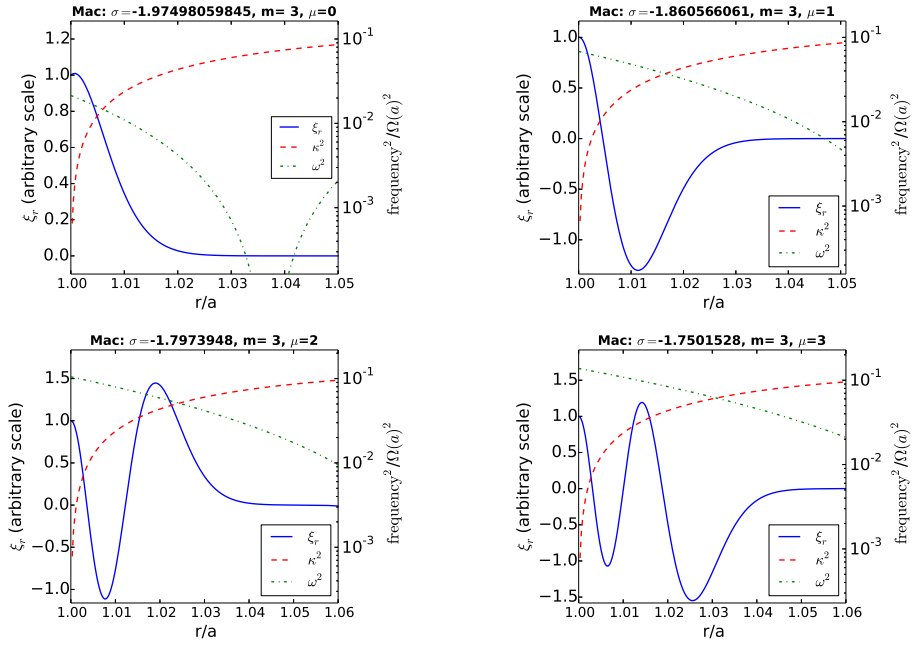


Figure 10. Same as Figure 6, but for $m = 3$.

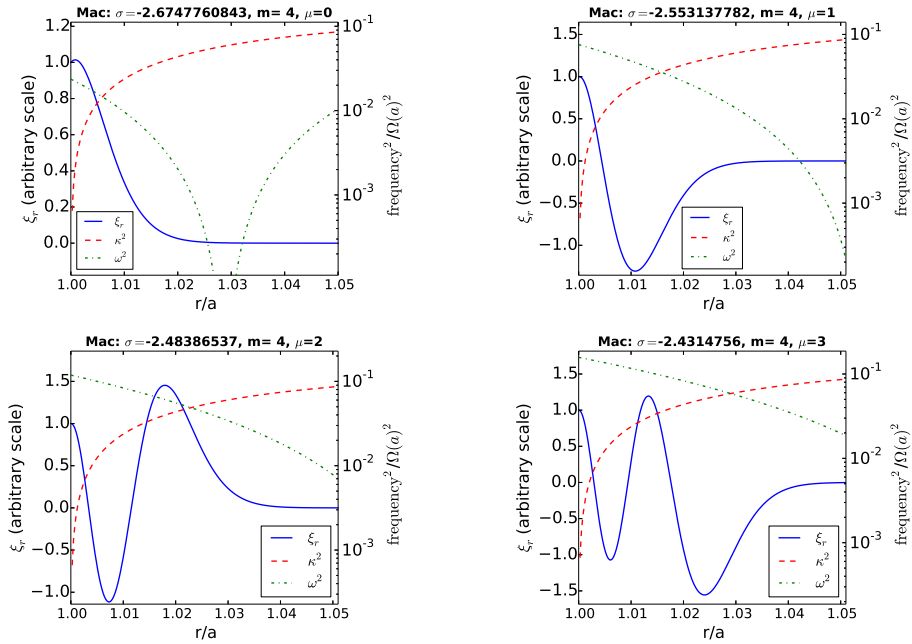


Figure 11. Same as Figure 6, but for $m = 4$.

the previous sections for the black-hole disk. For both the GR (“KL”) and the Newtonian (Maclaurin) $m = 0$ model the ratio of the $\mu = 2$ frequency to the fundamental is very close to 2:1.

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