Stress-energy tensor of a radiating sphere inclosing black hole

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ABSTRACT

We consider a uniformly luminous radiating sphere and a static black hole located in the center of that sphere. We give analytic formulas for radiation stress-energy tensor components in such a configuration, for the observer located at an arbitrary distance from the static black hole horizon.

Keywords: black hole – radiation in general relativity – stress-energy tensor

1 INTRODUCTION

In Abramowicz et al. (1990) an analytic formula for the stress-energy tensor of uniformly radiating static relativistic star was found, for ZAMO (zero angular momentum observer) at the arbitrary distance from the star surface. The result is very useful for investigations of the test particle motion in a curved spacetime, under the radiation four-force influence and several groups pursue this branch of research in recent years, see e.g. Bini and Geralico (2010); Sok Oh et al. (2011); Stahl et al. (2012); Wielgus et al. (2012); Stahl et al. (2013). Here we consider the extended problem, i.e. the observer may be located inside the uniformly radiating luminous sphere of radius R. In that way we complete the solution of Abramowicz et al. (1990), allowing for any ratio of the radiating sphere's and observer's radii in the Schwarzschild spacetime (it is assumed that the radiating sphere does not contribute to the spacetime curvature). But the problem is not only interesting for the reason of the mathematical completeness of this solution. Such a solution is particularly well suited to describe the interactions between black hole and the Cosmic Background Radiation, which corresponds to the uniformly luminous sphere located at infinity. The extended solution is also physically relevant when we consider the radiation from numerous, randomly spread sources, that can be approximated by a homogeneous luminosity. That can be the case of a black hole located in the center of a spherical galaxy, for instance, or the radiatively efficient spherical accretion on the black hole.

2   CASE OF THE LUMINOUS INNER SPHERE

Let us first review the solution for an observer located above the static luminous sphere in Schwarzschild spacetime, given by Abramowicz et al. (1990). We follow a slightly different approach, giving some more general formulas, in order to make a very smooth extension to the case of $r < R$ (observer located at radius r inside the luminous sphere of radius R) in the next section. Assuming homogeneous, isotropic radiation flux in the emitter's rest frame we conclude that the luminosity observed by any static observer located at a given radial location is constant over the observed sphere surface. Thus, the problem of calculating the radiation stress-energy tensor components in the static ZAMO frame reduces to the calculation of the constant specific intensity moments. Hence, radial dependence of the following two quantities need to be established

(1)   specific intensity $I(r)$ (value to be integrated over the observer's local sky),
(2)   sphere viewing angle α_0 (boundary for the specific intensity integration).

Specific intensity as seen by the observer located at radial distance r corresponds to the surface intensity $I(R)$ of the radiation source, gravitationally redshifted by the presence of the central mass. The quantities that are conserved along the particular light ray travelling through the curved spacetime are the *photon energy*

$$E = p^\alpha \eta_\alpha = 0 = \text{const.} , \tag{1}$$

for the photon four-momentum p^α and Killing vector $\eta^\alpha = \delta^\alpha_t$, and the *redshifted intensity*

$$I_0 = \frac{I(r)}{\left(p^\alpha v_\alpha\right)^4} = \text{const.} , \tag{2}$$

for the stationary observer's four-velocity $v^\alpha = \eta^\alpha (|g_{tt}|)^{-1/2}$, see Misner et al. (1973) section 4.22 for some more details. Hence, we find

$$I(r) = I(R) \left(\frac{1 - 2M/R}{1 - 2M/r} \right)^2 , \tag{3}$$

so clearly $I(r) < I(R)$ for $r > R$. Viewing angle α_0 corresponds to the largest possible value of angle α in the Fig. 1, which occurs for the largest possible emission angle δ_0 for which the photon can be observed. For $R \geq 3M$ $\delta_0 = \pi/2$, i.e. all emitted photons are able to escape from the star vicinity. This is not true for $2M < R < 3M$, where

$$\sin^2 \delta_0 = 27(1 - 2M/R)/R^2 . \tag{4}$$

This result will be explained a little further. The relevant angles can be defined using the photon four-velocity in the local orthonormal frame $u^{(\alpha)}$, i.e.

$$\tan \alpha = \left[\frac{u^{(\varphi)}}{u^{(r)}} \right]_{\text{OB}} = \left[\frac{g_{\varphi\varphi}\left(u^\varphi\right)^2}{g_{rr}\left(u^r\right)^2} \right]^{1/2}_{\text{OB}} \quad ; \quad \tan \delta = \left[\frac{u^{(\varphi)}}{u^{(r)}} \right]_{\text{EM}} = \left[\frac{g_{\varphi\varphi}\left(u^\varphi\right)^2}{g_{rr}\left(u^r\right)^2} \right]^{1/2}_{\text{EM}} , \tag{5}$$
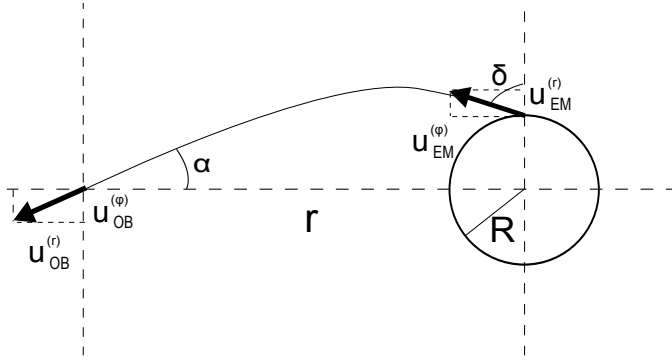
Figure 1. Photon emitted from the star surface R with the emission angle δ.

lower subscripts OB and EM denoting the location of the photon emission and observation, respectively. Using the normalization $u^\alpha u_\alpha = 0$ it is easy to show that the following equation *always* holds

$$\frac{r^2 \sin^2 \gamma}{1 - 2M/r} = -\frac{u_\varphi}{u_t} = \ell = \text{const.}, \tag{6}$$

where $\tan \gamma = u^{(\varphi)}/u^{(r)}$, $\gamma_{\text{OB}} = \alpha$ and $\gamma_{\text{EM}} = \delta$, from which we find that

$$\sin^2 \alpha_0 = \left(\frac{R}{r}\right)^2 \left(\frac{1 - 2M/r}{1 - 2M/R}\right) \sin^2 \delta_0 \tag{7}$$

and since $\alpha_0(r)$ must decrease with radius, the *only* solution is

$$\alpha_0(r) = \begin{cases} \arcsin\left[\frac{R}{r}\left(\frac{1-2M/r}{1-2M/R}\right)^{1/2}\right] & \text{for } 3M \leq R \leq r \\[2ex] \arcsin\left[\frac{3\sqrt{3}M(1-2M/r)^{1/2}}{r}\right] & \text{for } R < 3M < r \\[2ex] \pi - \arcsin\left[\frac{3\sqrt{3}M(1-2M/r)^{1/2}}{r}\right] & \text{for } R < r < 3M \end{cases} \tag{8}$$

Finally, having calculated the $I(r)$ and $\alpha_0(r)$ distributions, the stress-energy tensor ZAMO components are found by the integration over the observer's local sky

$$T^{(\alpha)(\beta)}(r) = I(r) \int n^{(\alpha)} n^{(\beta)} d\Omega; \qquad n^{(\alpha)} = p^{(\alpha)}/p^{(t)} \tag{9}$$

to give

$$T^{(t)(t)} = 2\pi I(r)(1 - \cos \alpha_0), \tag{10}$$

$$T^{(t)(r)} = \pi I(r) \sin^2 \alpha_0, \tag{11}$$

$$T^{(r)(r)} = \frac{2}{3}\pi I(r)\left(1 - \cos^3 \alpha_0\right),$$ (12)

$$T^{(\theta)(\theta)} = T^{(\varphi)(\varphi)} = \frac{1}{3}\pi I(r)\left(2 - 3\cos \alpha_0 + \cos^3 \alpha_0\right).$$ (13)

All other components are simply equal to zero.

3   LUMINOUS OUTER SPHERE

The case of the observer located inside the luminous sphere is similar to certain extent, since in the static spacetime photons may travel along the same null geodesic trajectory in both directions. Hence, the relation (3) holds all the same, only difference being that $I(r) > I(R)$ for $R > r$, i.e. radiation is now *blueshifted* in the static observer's frame. Equation (3) also ensures that the observer's local sky is uniformly bright with exception of the part occluded by the black hole. Hence, the remaining part is to calculate the angular diameter of the black hole as a function of radius. For the analogy with the previous case, we denote this quantity with $2\alpha_0$, see Fig. 2. Note that it follows from the Eq. (6) that photons *always* cross the horizon $r = 2M$ with angle $\gamma = 0$, i.e. perpendicularly to the horizon surface. Let us now investigate the relation (6) in more details. One may notice that for a photon trajectory to extend from $r_0 \gg 2M$ to the black hole horizon it is necessary that we are able to define a meaningful photon radial four-velocity component for every $r_0 > r > 2M$, i.e. if

$$u^r u_r > 0 \implies u_\varphi u^\varphi + u_t u^t < 0 \implies \ell^2 \leq -\frac{g_{\varphi\varphi}}{g_{tt}} \leq 27.$$ (14)

The number 27 is a value of a global maximum of $-g_{\varphi\varphi}/g_{tt}$ that occurs for $r = 3M$. This means that only photons with $\ell < 3\sqrt{3}$ fall into the black hole and putting the maximum value of ℓ into Eq. (6), we find Eq. (4) (remember that the outgoing trajectory in the case of inner luminous sphere corresponds to the ingoing trajectory in the outer luminous sphere case). Considering that α_0 must decrease monotonously with r, the *only* solution for $\alpha_0(r)$
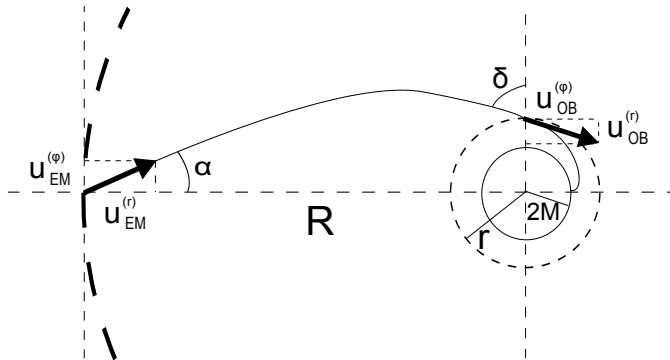
Figure 2. Photon emitted from the outer sphere surface R with the emission angle α.

that satisfies Eq. (6) is

$$
\alpha_0(r) = \begin{cases} \arcsin\left[\frac{3\sqrt{3}M(1-2M/r)^{1/2}}{r}\right] & \text{for } r \geq 3M \, , \\[3mm] \pi - \arcsin\left[\frac{3\sqrt{3}M(1-2M/r)^{1/2}}{r}\right] & \text{for } r < 3M \, , \end{cases} \tag{15}
$$

which is the same as the Eq. (8) in the case of inner radiating sphere radius $R < 3M$. Note that the formula for $\alpha(r)$ does not depend on luminous sphere radius R. Finally, after the local sky integration, we find

$$
T^{(t)(t)} = 2\pi I(r)(1 + \cos\alpha_0) \, , \tag{16}
$$

$$
T^{(t)(r)} = -\pi I(r)\sin^2\alpha_0 \, , \tag{17}
$$

$$
T^{(r)(r)} = \frac{2}{3}\pi I(r)\left(1 + \cos^3\alpha_0\right) \, , \tag{18}
$$

$$
T^{(\theta)(\theta)} = T^{(\phi)(\phi)} = \frac{1}{3}\pi I(r)\left(2 + 3\cos\alpha_0 - \cos^3\alpha_0\right) \, . \tag{19}
$$

This system is quite similar to the result of Abramowicz et al. (1990), yet the flux is of a different sign, and the angle is substituted $\alpha_0 \to \pi - \alpha_0$. It is interesting to observe, that $\alpha_0 \to \pi$ as $r \to 2M$, so the area of integration (bright sky region) goes to zero as the observer approaches the horizon. On the other hand, $I(r)$ given by Eq. (3) diverges in such a limit. So does the result of integration, the stress-energy tensor components, vanish in the limit of the horizon, diverge or have some finite limit? The answer to this question and its implications are discussed in details by Wielgus et al. (2014).

4   CONCLUSIONS

We presented the extension of the classic analytic calculation of a static luminous star radiation stress-energy tensor to the case of a luminous sphere observed from the inside. We found out that because of the symmetries involved, such a problem has a very similar solution. Analysis of the Cosmic Background Radiation field properties close to the black hole horizon is one example of application of the presented formulas.

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