New exact solutions for Schwarzschild-like black holes

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ABSTRACT
We use the so called mass function method for obtaining new exact solutions for Schwarzschild-like black holes. The mass function is algebraic invariant for general spherically symmetric metric. Rewriting Einstein equations in terms of mass function we consider Schwarzschild-like black holes. We show that Schwarzschild black holes should necessarily contain non-baryonic matter. We furthermore obtain new exact solutions for the black hole embedded into the dust matter universe.

Keywords: Scwarzschild-like lack hole – exact solutions of Einstein equations – black hole immersed in dust medium

1 INTRODUCTION

The mass function (MF) is algebraic invariant for spherically symmetric metric (Narlikar and Karmarkar, 1949). For the interval of general form

$$\mathrm{d}s^2 = e^{\nu(R,t)}\mathrm{d}t^2 - e^{\lambda(R,t)}\mathrm{d}R^2 - r^2(R,t)\mathrm{d}\sigma^2 \tag{1}$$

the MF is defined as follows

$$m(R,t) = r(R,t)\left(1 + e^{-\nu(R,t)}\dot{r}^2 - e^{-\lambda(R,t)}r'^2\right), \tag{2}$$

where prime means $\partial/\partial R$, and dot means $\partial/\partial t$; here and further $c = 1$, $\mathrm{d}\sigma^2$ is a standard metric on 2-sphere.

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Using MF it is possible to rewrite the Einstein equations in simpler way:

$$m' = \varepsilon r^2 r', \tag{3}$$
$$\dot{m} = -p_{\|} r^2 \dot{r}, \tag{4}$$
$$2\dot{r}' = \nu' \dot{r} + \dot{\lambda} r', \tag{5}$$
$$2\dot{m}' = m' \frac{\dot{r}}{r'} \nu' + \dot{m} \frac{r'}{\dot{r}} \dot{\lambda} - 4 r \dot{r} r' p_{\perp}. \tag{6}$$

where ε is energy density, including $8\pi\gamma/c^4$(so that it has a dimension of cm^{-2}), $p_{\|}$ is radial pressure, $p_{\perp}$ is tangent pressure ($T_2^2 = -p_{\perp}$) also in cm^{-2}. In the system (3)–(6) the equation (3) corresponds to the equation for T_0^0: $R_0^0 - R/2 = \varepsilon$, the equation (4) corresponds to the one for T_1^1, the equation (5), is in fact a condition of comoving frame $T_0^1 = 0$, the equation (6) follows from the equation for T_2^2.

2 SOLUTIONS FOR THE SCHWARZSCHILD-LIKE BLACK HOLES

Here we consider the class of static spherically-symmetric solutions in curvature coordinates (Gautreau, 1984):

$$\mathrm{d}s^2 = e^{\nu(r)}\mathrm{d}t^2 - e^{\lambda(r)}\mathrm{d}r^2 - r^2\mathrm{d}\sigma^2. \tag{7}$$

For this metric MF reads

$$m(r) = r\left(1 - e^{-\lambda(r)}\right). \tag{8}$$

And the energy density in this case according to (3) will be

$$\varepsilon(r) = \frac{m'(r)}{r^2}. \tag{9}$$

Let us take anisotropic fluid as a source. Thus we have the stress-energy tensor with following nonzero components: $T_0^0 = \varepsilon$, $T_1^1 = -p_{\|}$, $T_2^2 = T_3^3 = -p_{\perp}$.

In order to obtain an exact solution for the case under consideration we will fix the coordinate conditions as follows:

$$e^{\nu(r)} = e^{-\lambda(r)}. \tag{10}$$

Taking into account (10) and (8) one can rewrite the metric (7)

(11)

Einstein equations for the components T_0^0 and T_1^1 under condition (10) will lead to the equality

$$T_0^0 = T_1^1, \tag{12}$$

that means that the radial pressure is always negative $p_{\|} = -T_1^1$. Thus one can conclude that under condition (10) the space-time (11) always contains non-baryonic matter.

Let us now consider the stress-energy conservation law $T^{\mu}_{\nu;\mu} = 0$ which is more suitable here than the rest equation for T^2_2. For the case under consideration one has

$$\frac{\partial}{\partial r} T^1_1 + \frac{2}{r} T^1_1 + \frac{2}{r} T^2_2 = 0, \tag{13}$$

here we take into account that $T^2_2 = T^3_3$ for the spherical symmetry. Substituting radial pressure expressed from (9) and (12) into (13) one has for the tangent pressure:

$$T^2_2 = \frac{m''(r)}{2r} = -p_{\perp}. \tag{14}$$

So all the metric coefficients and nonzero components of the stress-energy tensor are expressed in terms of the MF and its derivatives. The state equation in this case (under condition (12)) will be the relation between radial and tangent parts of pressure, and it will completely define the solution.

The metric (8) allows different types of solutions concerning their physical properties.

(1) Under $m(r) < r$ for all r ($0 < r < \infty$) these solutions describe the only R-region. As far as radial pressure is negative these solutions can possibly describe the spherical configurations with "dark energy".

(2) Under $m(r) > r$ for all r the solutions describe the T-region only. For example the state equation $T^2_2 = \frac{r}{2} T^1_1$ leads to the following T-solution:

$$\mathrm{d}s^2 = \left(\frac{r_0}{t} e^{\frac{t}{r_0}} - 1\right)^{-1} \mathrm{d}t^2 - \left(\frac{r_0}{t} e^{\frac{t}{r_0}} - 1\right) \mathrm{d}r^2 - t^2 \mathrm{d}\sigma^2, \tag{15}$$

where r_0 is an arbitrary constant.

(1) If the metric coefficients $e^{\nu(r)}$ and $e^{\lambda(r)}$ are alternating functions the solutions will describe the Schwarzschild-like black holes. Let us suppose that the following equality is valid

$$T^2_2 = \beta T^1_1, \tag{16}$$

where β is an arbitrary constant. From (9) and (14) under condition (16) one obtains the following equation for the MF

$$m''(r) - 2\beta m'(r) \frac{1}{r} = 0, \tag{17}$$

which has the solution

$$m(r) = C_1 \frac{r^{2\beta+1}}{2\beta + 1} + C_2. \tag{18}$$

Let us choose the constants to be $C_2 = r_g$ ($r_g = \frac{2\gamma M}{c^2}$ is Schwarzschild radius), $\frac{C_1}{2\beta+1} = A$ ($2\beta + 1 \neq 0$), and rewrite the MF

$$m(r) = r_g + A r^{2\beta+1}. \tag{19}$$

$\beta = 1$ in (19) gives either de Sitter solution under $r_g = 0$ or the Kottler solution under nonzero r_g. If $\beta = -1$ there will be Reissner-Nordstrem solution. If $\beta = \frac{1}{2}$ one will have the solution obtained in (Mannheim, 2006; Grumiller, 2010) which describe the Rindler space-time which is actively discussed last time concerning the interior of black holes (Culetu, 2012a,b). The case of $\beta = -\frac{1}{2}$ is a special, it gives $e^{-\lambda} = 1 - \frac{r_g}{r} - \frac{r_0}{r}\ln\frac{r}{r_0}$.

For all $\beta \neq -\frac{1}{2}$ it is possible to write the solution in general form

$$\mathrm{d}s^2 = \left(1 - \frac{r_g}{r} - \sum_i A_i r^{2\beta_i}\right)\mathrm{d}t^2 - \left(1 - \frac{r_g}{r} - \sum_i A_i r^{2\beta_i}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\sigma^2. \tag{20}$$

Here we take into account the additivity of MF. If one has several sources, their contributions to the metric coefficients in (20) will be summarised (under supposition of weak interaction between the sources $\varepsilon = \sum_i \varepsilon_i$, $p = \sum_i p_i$)

The physical sense of different β is not obvious. But in (Fernando, 2012) the black hole was considered surrounded by quintessence that corresponds to $\beta = -\frac{1}{3}$. The problem of investigation of the general properties of such black holes is a question of our further great interest.

3 THE SOLUTIONS FOR THE BLACK HOLES ON THE DUST MATTER BACKGROUND

The problem of building a model of the black hole that embedded into space which is not empty but filled with some matter is of great interest in wide set of research directions.

One of the pioneer works in this direction is the paper by McVittie (McVittie, 1933) were the solution was obtained in the following form

$$\mathrm{d}s^2 = \left[\frac{1 - r_g\mu(t)/4R}{1 + r_g\mu(t)/4R}\right]^2 \mathrm{d}t^2 - \frac{1}{\mu^2(t)}\left[1 + \frac{r_g\mu(t)}{4R}\right]^4 \left(\mathrm{d}R^2 + R^2\mathrm{d}\sigma^2\right). \tag{21}$$

If $\mu(t) = \mathrm{const} = 1$ the expression (21) will give the Schwarzschild metric in isotropic coordinates. Under $r_g = 0$ the metric (21) takes the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{1}{\mu^2(t)}\left(\mathrm{d}R^2 + R^2\mathrm{d}\sigma^2\right), \tag{22}$$

that can be treated as a Friedman model for the flat space case. Depending on the choice of $\mu(t)$ it can be either dust model or the model with nonzero pressure. McVittie proposed his solution pretending to be a model of the massive particle in the expanding universe. But in papers (Nolan, 1998; McClure and Dyer, 2006) it was shown that this solution is unsuitable for the point mass in the universe, although it may possibly be used to describe the black hole.

To study the physical point of the solution (21) we will find the expression for the MF for it:

$$m(R,t) = r_g + R^3\frac{\dot{\mu}^2(t)}{\mu^5(t)}\left(1 + \frac{r_g}{4R}\mu(t)\right)^6. \tag{23}$$

From here one has $m = r_g$ under $\dot{\mu}(t) = 0$, i.e. the MF for the Schwarzschild solution. And under condition $r_g = 0$ and $\frac{\dot{\mu}^2(t)}{\mu^5(t)} = \mathrm{const} \equiv \frac{1}{a_0^2}$ one has the MF for the Friedman solution for the dust. Let us find using 3 and 4 the energy density and pressure for the metric (21):

$$
\begin{aligned}
\varepsilon(t) &= 3\frac{\dot{\mu}^2(t)}{\mu^2(t)},\\
p(R,t) &= 6\mu^3(t)\frac{r_g/4R}{\left(r_g/4R\right) - 1}.
\end{aligned}
\tag{24}
$$

From the expressions (23) and (24) it is clear that the solution (21) can describe neither point mass nor the black hole in the Friedman space-time (as far as pressure depends on R and t).

The solutions that were obtained in papers (Faraoni and Jacques, 2009; Thakurta, 1981; Sultana and Dyer, 2005), are in fact the modifications of the solution (21). So the solution from (Faraoni and Jacques, 2009) is (21) with metric coefficients near dR^2 and $d\sigma^2$ multiplied by some function of time. The solution from (Thakurta, 1981; Sultana and Dyer, 2005) is (21) with $\mu(t) = \mathrm{const}$ and metric coefficients near dR^2 and $d\sigma^2$ also multiplied by function of time. Analogous consideration of these solutions using the MF method shows that they also describe neither point mass nor the black hole in the Friedman or even Tolman-Bondi space-time because the pressure in them depends both on R and t.

Using the MF method we will now build the model for the black hole embedded into Tolman-Bondi universe.

Tolman-Bondi metric for the spherically-symmetric dust distribution in comoving frame (which is synchronous for the dust) has the form:

$$
\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{r'^2(R,t)}{f^2(R)}\mathrm{d}R^2 - r^2(R,t)\mathrm{d}\sigma^2,
\tag{25}
$$

Three types of the Tolman solution are well known:

Hyperbolic ($f^2(R) > 1$):

$$
\begin{aligned}
r(R,t) &= \frac{m(R)}{f^2(R) - 1}\sinh^2\frac{\alpha}{2},\\
t - t_0(R) &= \pm\frac{m(R)}{2(f^2(R) - 1)^{3/2}}(\sinh\alpha - \alpha).
\end{aligned}
\tag{26}
$$

Elliptic ($f^2(R) < 1$):

$$
\begin{aligned}
r(R,t) &= \frac{m(R)}{1 - f^2(R)}\sin^2\frac{\alpha}{2},\\
t - t_0(R) &= \frac{m(R)}{2(1 - f^2(R))^{3/2}}(\alpha - \sin\alpha).
\end{aligned}
\tag{27}
$$

Parabolic($f^2(R) = 1$):

$$r(R,t) = \left[\pm\frac{3}{2}\sqrt{m(R)}(t - t_0(R))\right]^{\frac{2}{3}}. \tag{28}$$

$m(R)$, $f(R)$ and $t_0(R)$ are three arbitrary functions of the solutions. $f(R)$ has the meaning of total energy in mc^2 units in the shell $R = \text{const}$; $m(R)$ is a total mass in the shell $R = \text{const}$:

$$m = \int_0^R \frac{8\pi\gamma}{A^4}\varepsilon(R,t)r^2(R,t)\frac{\partial r}{\partial R}\mathrm{d}R. \tag{29}$$

The Schwarzschild solution is a particular case of the Tolman-Bondi solution under $m(R) = r_g$. The Friedman solution comes from the Tolman-Bondi solution under certain choice of arbitrary functions, for example for the flat case one has $m(R) = a_0R^3$, $f(R) = 1$, $t_0(R) = 0$.

Thus the solution for the Schwarzschild-like black hole in the Tolman-Bondi space-time will be the Tolman-Bondi solution with combined MF:

$$m(R) = r_g + m_F(R). \tag{30}$$

Thus the solution for the flat Friedman world with embedded Schwarzschild black hole will have the form

$$r(R,t) = \left[\pm\frac{3}{2}\sqrt{r_g + a_0R^3}(t - t_0(R))\right]^{\frac{2}{3}}. \tag{31}$$

One should notice that (31) does not describe the pure Friedman world. This is the Tolman-Bondi world with the same MF as in the Friedman solution because $t_0(R)$ can not be zero for the flat Schwarzschild solution (unlike $t_0(R) = 0$ for Friedman).

The solution obtained in (Changjun, 2011)

$$r(R,t) = \left[\frac{3}{2}\left(\sqrt{a_0R^3} + \sqrt{r_g}\right)t + R^{\frac{3}{2}}\right]^{\frac{2}{3}}. \tag{32}$$

has in round brackets the sum of two correspondent expressions for the Friedman and Schwarzschild space-time with chosen $t_0(R) = -R^{3/2}a_0$. The MF for this solution reads

$$m(R) = a_0R^3 + r_g + 2\sqrt{a_0r_g}R^{3/2}. \tag{33}$$

And thus this is black hole in some specific space-time with MF (33) but not in universe filled with dust.

Supposing in (27) $f(R) = \cos R$ as for the closed Friedman solution and taking MF as $m(R) = r_g + a_0\sin^3 R$ one has

$$r(R,t) = \frac{r_g + a_0\sin^3 R}{\sin^2 R}\sin^2\frac{\alpha}{2},$$

$$t - t_0(R) = \frac{r_g + a_0\sin^3 R}{2\sin^3 R}(\alpha - \sin\alpha). \tag{34}$$

Similarly for (26) choosing $f(R) = \cosh R$ and $m(R) = r_g + a_0 \sinh^3 R$ one obtains

$$r(R,t) = \frac{r_g + a_0 \sinh^3 R}{\sinh^2 R} \sinh^2 \frac{\alpha}{2},$$
$$t - t_0(R) = \frac{r_g + a_0 \sinh^3 R}{2 \sinh^3 R}(\sinh \alpha - \alpha). \tag{35}$$

So by means of the MF method we have obtained new exact solutions (32), (34), (35) for the black hole on the dust background for the flat, closed and open cases respectively.

4 CONCLUSIONS

In this paper exact solutions for the empty space were considered from the point of view of the mass function method. The generalized solutions for the Schwarzschild-like black holes were obtained. It was shown that such black holes necessarily contain non-baryonic matter.

The problem of constructing the model of the black hole in non-empty space was investigated. It was shown why the known solutions pretending to describe such model are not consistent with the model itself. New exact solutions for the black hole embedded into the dust universe were obtained.

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