On the metric bundles of axially symmetric spacetimes

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ABSTRACT
We present the definition of metric bundles (**MBs**) in axially symmetric geometries and give explicit examples for solutions of Einstein equations. These structures have been introduced in Pugliese and Quevedo (2019) to explain some properties of black holes (**BHs**) and naked singularities (**NSs**), investigated through the analysis of the limiting frequencies of stationary observers, which are at the base of a Killing horizon definition for these black hole spacetimes. In Pugliese and Quevedo (2019), we introduced the concept of **NS** Killing throats and bottlenecks associated to, and explained by, the **MBs**. In particular, we proved that the horizon frequency can point out a connection between **BHs** and **NSs**. We detail this definition in general and review some essential **MBs** properties as seen in different frames and exact solutions.

Keywords: black holes – naked singularities – Killing horizons – metric bundles

1 INTRODUCTION

The aim of this work is to discuss the main properties of the metric bundles (**MBs**) for axially symmetric spacetimes, concentrating on some exact solutions. In this particular case, **MB** is a family of spacetimes defined by one characteristic photon (circular) orbital frequency ω and characterized by a particular relation between the metric parameters. This concept is used to establish a relation between black holes (**BHs**) and naked singularities (**NSs**) spacetimes. In Pugliese and Quevedo (2019), we performed an analysis of the **MBs** corresponding to the equatorial plane of the Kerr, Reissner-Nordström and Kerr-Newman geometries. The off-equatorial case of the Kerr spacetime is considered in detail in Pugliese and Quevedo (2019a).

A **MB** is represented by a curve on the so-called extended plane (Pugliese and Quevedo, 2019), which is the entire collection of a parameterized family of solutions. For concreteness, we now consider the family of Kerr spacetimes. All the **MBs** are tangent to the

978-80-7510-433-5

horizon curve as represented on the extended plane. Then, the horizon curve emerges as the envelope surface of the set of **MBs**. It turns out that **WNSs** (weak naked singularities), for which the spin-mass ratio is close to the value of the extreme **BH**, are related to a portion of the inner horizon, whereas strong naked singularities **(SNSs)** with $a > 2M$ are related to the outer horizon. In addition, **WNSs** are characterized by the presence of Killing bottlenecks, which are defined as "restrictions" of the Killing throats that appear in **WNSs**. Killing throats or tunnels, in turn, emerge through the analysis of the radii $r_s^{\pm}(\omega, a)$ of light surfaces, which depend on the frequency of the stationary observers ω and the spin parameter a (Pugliese and Quevedo, 2018, 2019). In the case of **NS** geometries, a Killing throat is a connected region in the $r - \omega$ plane, which is bounded by the radii $r_s^{\pm}(\omega, a)$ and contains all the stationary observers allowed within the limiting frequencies $[\omega_-, \omega_+]$. In the case of **BHs**, a Killing throat is either a disconnected region in the Kerr spacetime or a region bounded by non-regular surfaces in the extreme Kerr **BH** spacetime. The limiting case of a Killing bottleneck occurs in the extreme Kerr spacetime, as seen in the Boyer-Lindquist frame, where the narrowing actually closes on the **BH** horizons. Killing throats and bottlenecks were grouped in Tanatarov and Zaslavskii (2017) in structures named "whale diagrams" of the Kerr and Kerr-Newman spacetimes–see also Mukherjee and Nayak (2018); Zaslavskii (2018). Moreover, Killing bottlenecks, interpreted in Pugliese and Quevedo (2019) as "horizons remnants" and related to **MBs** in Pugliese and Quevedo (2019); Pugliese and Quevedo (2019a), appear also connected with the concept of pre-horizon regime introduced in de Felice (1991); de Felice and Usseglio-Tomasset (1991). The pre-horizon was analyzed in de Felice and Usseglio-Tomasset (1991). It was concluded that a gyroscope would conserve a memory of the static or stationary initial state, leading to the gravitational collapse of a mass distribution (de Felice and Usseglio-Tomasset, 1992; de Felice and Yunqiang, 1993; de Felice and Sigalotti, 1992; Chakraborty et al., 2017).

More in general, **MBs** have interesting properties that allow us to explore in an alternative way some aspects of the geometries that define the bundle, providing an alternative interpretation of Killing horizons (in terms of a set of solutions–the extended plane) and establishing a connection between **NSs** and **BHs**, based on the fact that each bundle is tangent to the horizon curve. Moreover, as we shall see below, metric bundles highlight some properties of the horizons that could influence the exterior properties of **BH** geometries by means of characteristic frequencies. The **MBs** concept can have significant repercussions in the study of **BH** physics, in the interpretation of **NSs** solutions and in the horizons and **BH** thermodynamics.

In this work, we present the **MBs** definition and discuss their properties in the context of **BH** thermodynamics. We analyze the Kerr, Kerr-Newman and Reissner-Nordström metric bundles. The explicit expressions for metric bundles in the Kerr-de Sitter spacetime are also given. Finally, we present some concluding remarks.

2 METRIC BUNDLES

We start by considering the case of the Kerr spacetime. The Kerr metric, in Boyer-Lindquist (BL) coordinates, can be expressed as

$$\begin{aligned} ds^2 =& -\frac{\Delta - a^2\sin^2\theta}{\rho^2}dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta\left(\left(a^2+r^2\right)^2 - a^2\Delta\sin^2\theta\right)}{\rho^2}d\phi^2 \\ & - 2\frac{aM\sin^2(\theta)\left(a^2-\Delta+r^2\right)}{\rho^2}d\phi dt\,, \end{aligned} \tag{1}$$

$$\Delta \equiv r^2 - 2Mr + a^2, \quad \text{and} \quad \rho^2 \equiv r^2 + a^2\cos^2\theta\,. \tag{2}$$

It describes an axisymmetric, stationary, asymptotically flat spacetime. The parameter $M \geq 0$ is interpreted as the mass of the gravitational source, while the rotation parameter $a \equiv J/M$ (spin) is the specific angular momentum, and J is the total angular momentum of the source. This is a stationary and axisymmetric geometry with Killing fields $\xi_t = \partial_t$ and $\xi_\phi = \partial_\phi$, respectively.

In this work, we will consider also the Kerr-Newman (KN) geometry which corresponds to an electrovacuum axisymmetric solution with a net electric charge Q, described by metric (1) with $\Delta_{KN} \equiv r^2 + a^2 + Q^2 - 2Mr$. The solution $a = 0$ and $Q \neq 0$ constitutes the static case of the spherically symmetric and charged Reissner-Nordström spacetime. The horizons and the outer and inner static limits for the **KN** geometry are, respectively,

$$r_\mp = M \mp \sqrt{M^2 - (a^2+Q^2)}, \quad r_\epsilon^\mp = M \mp \sqrt{M^2 - a^2\cos^2\theta - Q^2}\,, \tag{3}$$

which for $a = 0$, $Q = 0$, and $a = Q = 0$ leads to $(r_\pm, r_\epsilon^\pm)$ in the Reissner-Nordström, Kerr and Schwarzschild geometries, respectively. Note that the **KN** horizons $r_\pm$ can be re-parameterized for the total charge Q_T and its variation with respect to the parameter Q_T is exactly the same as for the corresponding radii $r_\pm$ in the **RN** or Kerr solution. This aspect will be significant in the study of the **MBs** dependence from the two charges (a, Q).

On the BHs horizons

For the analysis of some properties of the horizons, we focus for simplicity on the case $Q = 0$. Then, for the Kerr **BH** geometry the horizons and ergospheres radii are given by $r_\pm = M \pm \sqrt{M^2 - a^2}$ and $r_\epsilon^\pm = M \pm \sqrt{M^2 - a^2\cos^2\theta}$, respectively.

Metric bundles are defined as the set of metrics that satisfy the condition $\mathcal{L}_N \equiv \mathcal{L}\cdot\mathcal{L} = 0$, where $\mathcal{L}$ is the Killing vector $\mathcal{L} \equiv \partial_t + \omega\partial_\phi$. Solutions could be either **BHs** or **NSs**. The quantity ω will be called the frequency or the **MBs** angular velocity. In **BH** spacetimes, this Killing vector defines also the thermodynamic variables and the Killing horizons.

On the Killing vector $\mathcal{L}$ and the condition $\mathcal{L}_N = 0$

The event horizons of a spinning **BH** are Killing horizons with respect to the Killing field $\mathcal{L}_H \equiv \partial_t + \omega_H\partial_\phi$, where ω_H is the angular velocity of the horizons, representing the **BH** rigid rotation (the event horizon of a stationary asymptotically flat solution with matter satisfying suitable hyperbolic equations is a Killing horizon). The Kerr horizons are, therefore, null surfaces, $\mathcal{S}_0$, whose null generators coincide with the orbits of an one-parameter group of isometries, i.e., in general, there exists a Killing field $\mathcal{L}$, which is normal to $\mathcal{S}_0$. In general, a Killing horizon is a lightlike hypersurface (generated by the flow of a Killing vector), where the norm of a Killing vector is null. In the limiting case of the static Schwarzschild spacetime ($a = 0$, $Q = 0$) or the Reissner Nordström spacetime ($a = 0$, $Q \neq 0$), the event horizons are Killing horizons with respect to the Killing vector ∂_t. More precisely, for static (and spherically symmetric) **BH** spacetimes, the event, apparent, and Killing horizons with respect to the Killing field ξ_t coincide.

The **BH** event horizon of stationary solutions have constant surface gravity (which is the content of the zeroth **BH** law-area theorem– the surface gravity is constant on the horizon of stationary black holes (Chrusciel et al., 2012; Wald, 1999)). The **BH** surface area is non-decreasing (second **BH** law establishing the impossibility to achieve by a physical process a **BH** state with zero surface gravity.) Moreover, the **BH** surface gravity, which is a conformal invariant of the metric, may be defined as the rate at which the norm $\mathcal{L}_\mathcal{N}$ of the Killing vector $\mathcal{L}$ vanishes from outside ($r > r_+$). (For a Kerr spacetime, this is $\mathcal{SG}_{Kerr} = (r_+ - r_-)/2(r_+^2 + a^2)$ and, however, the surface gravity re-scales with the conformal Killing vector, i.e. it is not the same on all generators but, because of the symmetries, it is constant along one specific generator). In the extreme case, where $r_\pm = M$, the surface gravity is zero and, consequently, the temperature is $T_H = 0$, but its entropy (and therefore the **BH** area) is not null (Chrusciel et al., 2012; Wald, 1999, 2001). This fact has consequences also with respect to the stability against Hawking radiation (a non-extremal **BH** cannot reach an extremal case in a finite number of steps–third **BH** law). The variation of the **BH** mass, horizon area and angular momentum, including the surface gravity and angular velocity on the horizon, are related by the first law of **BH** thermodynamics: $\delta M = (1/8\pi)\kappa\delta A + \omega_H \delta J$. In here, the term dependent on the **BH** angular velocity represents the "work term" of the first law, while the fact that the surface gravity is constant on the **BH** horizon, together with other considerations, allows us to associate it with the concept of temperature. More precisely, we can formalize this relation by writing explicitly the Hawking temperature as $T_H = \hbar c\kappa/2\pi k_B$, where k_B, is the Boltzmann constant and κ is the surface gravity. Temperature $T = \kappa/(2\pi)$; entropy $S = A/(4\hbar G)$, where A is the area of the horizon $A = 8\pi m r_+$; pressure $p = -\omega_h$; volume $V = GJ/c^2$ ($J = amc^3/G$); internal energy U= GM ($M = c^2 m/G$= mass), where m is the mass.

Here we study **MBs** which are defined by the condition $\mathcal{L}_\mathcal{N} = 0$; therefore, it is convenient to re-express some of the concepts of **BH** thermodynamics mentioned before in terms of $\mathcal{L}_\mathcal{N}$. Firstly, the norm $\mathcal{L}_\mathcal{N} \equiv \mathcal{L}^\alpha \mathcal{L}_\alpha$ is constant on the **BH** horizon. Secondly, the constant $\kappa : \nabla^\alpha \mathcal{L}_\mathcal{N} = -2\kappa\mathcal{L}^\alpha$, evaluated on the *outer* horizon r_+, defines the **BH** surface gravity, i.e., κ =constant on the orbits of $\mathcal{L}$ (equivalently, it is valid that $\mathcal{L}^\beta \nabla_\alpha \mathcal{L}_\beta = -\kappa \mathcal{L}_\alpha$ and $L_\mathcal{L}\kappa = 0$, where $L_\mathcal{L}$ is the Lie derivative–a non affine geodesic equation).

Stationary observers and causal structure

The condition $\mathcal{L}_\mathcal{N} = 0$ is also related to the definition of stationary observers. Stationary observers are characterized by a four-velocity of the form $u^\alpha = \gamma\mathcal{L}^\alpha$ ($\mathcal{L}^\alpha \equiv \xi_t^\alpha + \omega\xi_\phi^\alpha$); thus, $\gamma^{-2} \equiv -\bar{\kappa}\mathcal{L}_\mathcal{N}$, where γ is a normalization factor. The spacetime causal structure of the Kerr geometry can be then studied by considering also stationary observers (Malament, 1977): *timelike* stationary particles have limiting orbital frequencies, which are the photon orbital frequencies $\omega_{\pm}$, i.e., solutions to the condition $\mathcal{L}_{N} = 0$:

$$\omega_{\pm} \equiv \omega_Z \pm \sqrt{\omega_Z^2 - \omega_*^2}, \quad \omega_*^2 \equiv \frac{g_{tt}}{g_{\phi\phi}} = \frac{g^{tt}}{g^{\phi\phi}}, \quad \omega_Z \equiv -\frac{g_{\phi t}}{g_{\phi\phi}}. \tag{4}$$

Therefore, timelike stationary observers have orbital frequencies (from now on simply called frequencies) in the interval $\omega \in [\omega_-, \omega_+]$. Thus, frequencies $\omega_{\pm}$ evaluated on $r_{\pm}$ provide the frequencies $\omega_H^{\pm}$ of the Killing horizons.

For completeness, we also derive the frequencies ω_H of the horizons in the Kerr-Newman case,

$$\omega_H^- = \frac{aM\left(2M\sqrt{M^2-(a^2+Q^2)} - Q^2 + 2M^2\right)}{4M^2a^2 + Q^4}, \tag{5}$$

$$\omega_H^+ = \frac{aM}{2M\sqrt{M^2-(a^2+Q^2)} - Q^2 + 2M^2}. \tag{6}$$

The limiting Reissner-Nordström and Kerr cases can be obtained by imposing the conditions $a = 0$ and $Q = 0$, respectively.

Metric bundles: definition, structure and characteristic frequencies

Metric bundles are a set of metric tensors that can include only **BHs** *or* **BHs** *and* **NSs**, such that each geometry of the set has, at a certain radius r, equal limiting photon frequency $\omega_b \in \{\omega_+, \omega_-\}$, which is called *characteristic bundle frequency*. Therefore, **MBs** are solution of the zero-norm condition $\mathcal{L}_N(\omega_b) = 0$.

It can be proved that *all* the **MBs** are tangent to the horizon curve in the extended plane[1]–see Fig. (1). Then, the horizon curve emerges as the envelope surface of the set of **MBs**. As a consequence, in Pugliese and Quevedo (2019) we introduced the concept of weak naked singularities (**WNSs**) as those metrics related to a portion of the inner horizon, whereas strong naked singularities (**SNSs**) are related to the outer horizon in the extended plane.

It can be proved that all the frequencies $\omega_{\pm}$, in any point of a **BH** or **NS** geometry, are horizon frequencies in the extended plane or, in other words, since the **MBs** are tangent to the horizon curve, each characteristic frequency of the bundle ω_b is a horizon frequency $\omega_b = \omega_H^x$, where $\omega_H^x \in \{\omega_H^-, \omega_H^+\}$.

For seek of clarity, first we formalize this definition for the Kerr case as a one-parameter family of solutions parameterized with the spin a (or a/M). The generalization to the case of several parameters is straightforward as, for example, in **KN** and **RN** geometries. These cases will be also addressed explicitly below. Particularly, the frequency ω_b of the bundle is the inner or outer horizon frequencies of the spacetime, which is tangent to the horizon at a radius r_g and a spin a_g (*bundle tangent spin*). In addition, the bundle is characterized by the frequency ω_0 of the *bundle origin*, i.e., the point $r = 0$ and $a = a_0$ in the extended plane. Thus, the **MBs** are all characterized by a frequency $\omega_b = \omega_H^x(a_g)$, where a_g is the

[1] An *extended plane* π^+ is the set of points $(a/M, Q)$, where Q is any quantity that characterizes the spacetime and depends on a. In general, the extended plane is an $(n+1)$-dimensional surface, where n is the number of independent parameters that enter Q (Pugliese and Quevedo, 2019).

bundle tangent spin, and the frequency ω_0 at $r = 0$, where $a = a_0$. The relation between a_0, a_g, r_g, and ω_b, significant for the bundle characterization, is particularly simple in the case of a spherically symmetric geometry or on the equatorial plane of an axisymmetric geometry. However, in general, the relation, involving also the **MB** origin a_0, depends on the plane $\sigma \equiv \sin^2\theta$ (Pugliese and Quevedo, 2019a).

MBs can be closed on the horizon. In Pugliese and Quevedo (2019), this property has been shown to be due to the rotation of the singularity: the curves, which define the **BH** horizons for the static **RN** case, can be are open; the analysis of the **KN** case represented in Pugliese and Quevedo (2019) shows the influence of the spin in the bending and separation into two families of curves on the equatorial plane. On the other hand, in Pugliese and Quevedo (2019a) we proved that, on planes with $\sigma < 1$, there can be open Kerr bundles. Then, **MBs** of axisymmetric spacetimes have a non-trivial extension corresponding to negative bundle frequencies $\omega_b < 0$. These **MBs** extensions, associated to characteristic frequencies $\omega_b = -\omega_H^{\pm}$ equal in magnitude to the horizon frequencies, clearly are not tangent to the horizon curve in the negative frequencies extension of the extended plane. However, these **MBs** branches are tangent to the horizon curve in the plane with positive frequencies $-\omega_b > 0$.

Horizon relations for Kerr geometries on the equatorial plane $\sigma = 1$

Horizons relations I:
origin frequencies: $\omega_0^{-1} \equiv a_0^{\pm}/M = \frac{2r_{\pm}(a_g)}{a_g} \equiv \omega_H^{-1}(a_g)$;
horizons frequencies: $\omega_H^{+}(r_g, a_g) = \omega_0 = M a_0^{-1}$, $\omega_H^{-}(r_g', a_g) = \omega_0' = M/a_0'$, where $r_g' \in r_-$ $(r_+ = r_g, r_- = r_g')$.
Horizons relations II:
There is $\omega_0' = \frac{1}{4\omega_0}$, $\omega_H^{+}\omega_H^{-} = \frac{1}{4}$ and $(a_0^{+}(a_g)a_0^{-}(a_g) = 4M^2)$, $a_0^{\pm}/M = \frac{2r_{\pm}(a_g)}{a_g}$–see Pugliese and Quevedo (2019).

In the Kerr **MBs**, the Killing vector $\mathcal{L}_N$ is a function of r, a and $\sigma \equiv \sin^2\theta$. The equatorial plane is a notable case, showing in many aspects similarities with the case of static limiting geometries, where $\mathcal{L}_N$ is a function of r and a, only.

Explicit form of the metric bundles
Here, we present explicit expressions for the KN **MBs** and their limits:
Kerr geometries-equatorial plane $\sigma = 1$**:**

$$a_{\omega}^{\pm}(r, \omega; M) \equiv \frac{2M^2\omega \pm \sqrt{r^2\omega^2\left[M^2 - r(r+2M)\omega^2\right]}}{(r+2M)\omega^2}, \tag{7}$$

KN geometries-equatorial plane $\sigma = 1$**:**

$$a_{\omega}^{\mp} = \frac{\mp\sqrt{r^4\omega^2\left\{\omega^2\left[Q^2 - r(r+2M)\right] + M^2\right\}} + \omega M(Q^2 - 2rM)}{\omega^2\left[Q^2 - r(r+2M)\right]}, \tag{8}$$

$$\text{or}\quad (Q_{\omega}^{\pm})^2 \equiv \frac{r\left\{\omega^2\left[a^2(r+2M) + r^3\right] - 4aM^2\omega - rM^2 + 2M^3\right\}}{(a\omega - M)^2}, \tag{9}$$

RN geometries:

$$(Q_{\omega}^{\pm})^2 = r\left(\frac{r^3}{M^2}\omega^2 - r + 2M\right). \tag{10}$$

In the **RN** geometries, the limiting frequencies are $\omega_{\pm} = \pm\frac{M\sqrt{Q^2+(r-2M)r}}{r^2}$. The **KN** frequencies $\omega_{\pm}$ do not depend explicitly on Q_T; this means that the electric and rotational parameters of the geometry play a different role in the solutions $\omega_{\pm}$=constant.

Explicitly, if we consider a surface $a_g(a_0; Q)$ of the tangent **MBs** spins in the case $a_0 \neq 0$, where Q is a parameter, we obtain

KN bundle origin spin-equatorial plane:

$$a_0 = \frac{2M^2 - Q^2 \mp 2M\sqrt{M^2 - (a^2 + Q^2)}}{a} \quad (r_{\mp}), \tag{11}$$

KN bundle tangent spin:

$$a_g^{\mp}(a_0) = \frac{a_0\left(2M^2 - Q^2\right) \mp 2M\sqrt{a_0^2\left(M^2 - Q^2\right) - Q^4}}{a_0^2 + 4M^2} \tag{12}$$

$$\text{where} \quad a_0 > a_L(Q) \equiv \sqrt{-\frac{Q^4}{Q^2 - M^2}} \quad \text{with} \quad Q^2 \in [0, M^2].$$

These functions are very important to derive a relation between **BHs** (with tangent spins a_g) and **NSs** (with origin spins a_0), as discussed in Pugliese and Quevedo (2019), and also the transformation laws for **BHs** in the extended plane, as explicitly shown in Pugliese and Quevedo (2019a).

The relation between **BHs** and **NSs** can be formalized by analyzing the function of the tangent spin $a_g(a_0)$ in terms of the **MB** origin a_0 as follows

Kerr geometry $\sigma = 1$:

$$\forall\, a_0 > 0, \quad a_g \equiv \frac{4a_0 M^2}{a_0^2 + 4M^2} \quad \text{where} \quad a_g \in [0, M] \quad \text{and} \quad \lim_{a_0\to 0} a_g = \lim_{a_0\to\infty} a_g = 0,$$

$a_g(a_0 = 2M) = M$. Alternatively, we can explicitly write the relation between the tangent spin and the radius as follows:

$$a_{tangent}(r) \equiv \frac{r(M - r_g) + M r_g}{\sqrt{-(r_g - 2M)r_g}} \tag{13}$$

$$\text{where} \quad r_g \in [0, 2M], \quad a_g = a_{\pm} : \quad \frac{r_g}{M} \equiv \frac{2a_0^2}{a_0^2 + 4M^2}.$$

Some general results from the study of metric bundles in the extended plane

We now summarize some general results obtained in Pugliese and Quevedo (2019); Pugliese and Quevedo (2019a). For simplicity, we focus on the equatorial plane of the Kerr geometry so that a **MB** can be represented as a curve on the plane (a, r).

Vertical lines r =constant in the extended plane

Vertical lines r =constant in the extended plane intersect specific **MBs**. First, on a point r, there is always a maximum of two intersections (limiting cases are on the horizon curve or on the origin $r = 0$ and $a_0 = 0$ or $r = 2M$ and $a_0 = 0$), which provide the two limit frequencies $\omega_\pm \equiv \{\omega_b, \omega_b^{'}\}$, corresponding to the two characteristic frequencies of the two **MBs**. These are also horizon frequencies $\omega_\pm \equiv \{\omega_b, \omega_b^{'}\} \equiv \{\omega_H^x(a_g), \omega_H^y(a_g^{'})\}$, respectively, where $(x, y) = \pm$ and a_g and $a_g^{'}$. They are the tangent spins of the two **MBs** with frequency ω_b and $\omega_b^{'}$, respectively. We clarify in Pugliese and Quevedo (2019a) the precise correspondence between $\{x, y, \pm\}$. In fact, these quantities are related to the notion of **BH** inner horizon confinement, discussed firstly in Pugliese and Quevedo (2019), and to the horizon replicas introduced in Pugliese and Quevedo (2019a). The **BH** inner horizon confinement is related to the notion of bottleneck as well. It is based on the fact that it is not possible to find a bundle outside the outer event horizon ($r > r_+$) in the plane (and for any geometry a) with a characteristic frequency equal to that of the inner horizon. This implies that outside the horizon of a given spacetime, it is not possible to find a photon limiting frequency equal to the inner horizon frequency. Nevertheless, it is possible to find such orbits for the frequencies of the outer horizon. However, it is possible to find frequencies of the inner horizon in the Kerr case for σ sufficiently small (sufficiently close to the rotation axis); therefore, it is possible to "extract" this inner horizon frequency on an "orbit" $r > r_+ : \mathcal{L} \cdot \mathcal{L} = 0$.

This notion led to the definition in Pugliese and Quevedo (2019a) of the horizon replicas. These structures occur when there is a point r of the bundle such that the characteristic bundle frequencies $\omega_b(a) \in \{\omega_H^+(a_p), \omega_H^-(a_p)\}$ are located exactly at $r_\pm(a_p) > r_+(a)$, that is, on the horizon with frequency $\omega_b(a)$. Such orbits are, therefore, called horizons replicas (these are clearly related to the vertical lines of the extended plane crossing the horizon curve on the tangent point to the bundle).

Horizontal lines a =constant on the extended plane

Horizontal lines a =constant on the extended plane determine a particular geometry and are related to the orbits with frequencies equal to that of the Killing horizons in the extended plane and, therefore, to the concept of horizon replicas.

The Kerr-de-Sitter metric bundle

To complete this overview of the **MBs** of axisymmetric spacetimes, we present here the explicit expressions for the Kerr-de Sitter geometry, which has an interesting and complex horizon structure. Further details on these specific solutions can be found in Pugliese and Stuchlík (2019b).

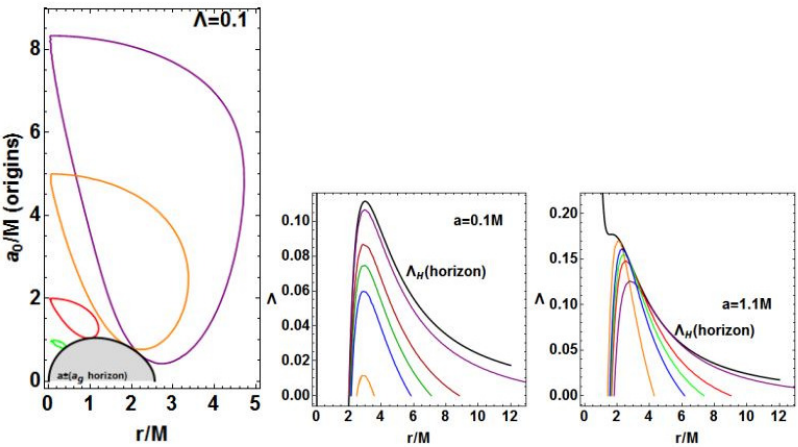

Figure 1. Kerr-de-Sitter geometry: Equatorial plane ($\sigma = 1$). Left panel: Metric bundles of the Kerr-de-Sitter geometries in the plane $(a/M, r/M)$ for fixed cosmological constant $\Lambda > 0$. The black thick curve is the horizon curve in the extended plane. Metric bundles are tangent to the horizon curve. The origin spins a_0 are also shown. The tangent spin a_g is on the horizon curve. Each bundle (curve) has a specific frequency (the lower bundle corresponds to greater frequencies due to the fact that the inner horizon frequency is always greater then the outer horizon frequency, a part in the extreme **BH** case), which is the horizon frequency of the point (a, r) of the bundle, particularly, at the origin $(a = a_0, r = 0)$ and tangent point $(a = a_g, r = r_g)$. Center and right panel: Bundles in the $(\Lambda, r/M)$ plane for different spins and frequencies. Black curves represent the horizon. Each curve is for a different fixed frequency ω (the lower the curve, the greater the frequency).

Kerr-de-Sitter metric bundle, general form in Λ:

$$\Lambda_\omega \equiv \frac{-6\sin^2(\theta)\left[\omega^2\left(a^2+r^2\right)^2+a^2-4aMr\omega\right]}{\left[a^2\cos(2\theta)+a^2+2r^2\right]\left[a\sin^2(\theta)\left[a\omega^2\left(a^2+r^2\right)-2\omega\left(a^2+r^2\right)+a\right]+r^2\right]} + \frac{6\left[a^2\omega^2\sin^4(\theta)\left[a^2+r(r-2M)\right]+a^2+r(r-2M)\right]}{\left[a^2\cos(2\theta)+a^2+2r^2\right]\left[a\sin^2(\theta)\left[a\omega^2\left(a^2+r^2\right)-2\omega\left(a^2+r^2\right)+a\right]+r^2\right]}. \quad (14)$$

This expression gives the form of the **MBs** in the Kerr-de-Sitter spacetime in terms of the cosmological constant $\Lambda > 0$ for any plane $\sigma \equiv \sigma^2\theta$. Similar solutions can easily be found in terms of a_ω. The extended plane is represented, however, a 3D space. In Figs. (1), we show different representations of this case.

3 CONCLUDING REMARKS

We discussed the concept of metric bundles of axially symmetric spacetimes. In Eqs. (7), (8) and (9) explicit expression of these bundles are given on the equatorial plane of the

Kerr geometries, Kerr-Newman spacetimes and for the spherically symmetric Reissner-Nordström spacetime. In Eq. (14), we present the expression for the Kerr-de-Sitter geometry. Figs (1) illustrate these **MBs** and their main features such as the origins a_0 and the tangent points a_g on the horizon curve in the extended plane, where the **MBs** are represented as curves. At the end of Sec. (2), we discussed some results concerning the general properties of the geometries defined by the bundles, as extracted from the analysis of these structures, such as the **BH** horizon confinement and horizon replicas. The issues discussed in this article refer to the study of Pugliese and Quevedo (2019), where the concept of metric bundle was first introduced and the definition of Killing throat and bottleneck for the Kerr, Kerr-Newman and Reissner-Nordström spacetimes were considered. In Pugliese and Quevedo (2019a), we present the general definition on an arbitrary plane of the Kerr geometry and give definition of horizon replicas. In a future work, we intend to generalize this study to other spacetimes (Pugliese and Stuchlík, 2019b) and investigate in detail the consequences for the **BH** thermodynamical properties as described in Sec. (2). Kerr-de Sitter **MBs** eventually face the problem of finding a convenient **MBs** parametrization and definition in spherically symmetric spacetimes. The **MBs** tangency with the horizons curves, characteristic of the axially symmetric spacetimes, reduces to an approximation for the static geometries, while an adaptation of the (conformal invariant) **MBs** definitions to the static case is possible. **MBs** utility lies in enlightening spacetime properties emerging in the extended plane, related to the local causal structure and **BH** thermodynamics such as the surface gravity, temperature and luminosity. The extended plane and metric bundles connect different points of one geometry and different geometries, providing a new frame of interpretation of these metrics families. Some spacetime properties can be detected by stationary observers and the light-like orbits in the region outside the **BH** horizon. In this sense, we mention the horizons confinement and the replicas. There is a replica when certain properties of a **BH** horizon are replicated in other points of the same or different spacetimes. There is also the vice versa effect called confinement, as we proved for a portion of the Kerr inner horizon curve. Significant for the transformations from one solution to another, **MBs** represent a global frame for the **BHs** analysis. Of direct astrophysical interest, **MBs**, read in terms of the light surfaces, relate many aspects of **BHs** physics, such as "**BH**" images, and several processes, which constrain energy extraction, such as the **BHs** jet emission and jet collimation, or regulate the Blandford-Znajek process. They also constraint accretion disks or the Grad-Shafranov equation for the force free magnetosphere around **BHs**.

ACKNOWLEDGEMENTS

D.P. acknowledges partial support from the Junior GAČR grant of the Czech Science Foundation No:16-03564Y.

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