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On magnetized orbits around Schwarzschild black hole

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ABSTRACT
We study circular orbits of magnetized particle around Schwarzschild black hole immersed in the uniform magnetic field. Despite the topic overlaps with the one of [Classical and Quantum Gravity **20**, 469 (2003)], our calculations complement it by correcting effective potential of the magnetized particle presented in it. As a rule, the effective potential is independent of the energy of test particle. We briefly demonstrate the formalism and present qualitative picture on the effect of the magnetic coupling parameter on innermost stable circular orbits.

Keywords: Schwarzschild spacetime – magnetized particle – innermost stable circular orbit

In the paper (de Felice and Sorge, 2003), the circular orbits of a particle with mass m possessing magnetic dipole momentum $\boldsymbol{\mu}$ around Schwarzschild black hole with mass M immersed in the asymptotically uniform magnetic field $\boldsymbol{B}$ that is perpendicular to the orbital plane was studied. In this paper we refine the results presented in that paper by addressing the problem associated with that effective potential of the magnetized particle. For the reader's sake, prior to attracting one's attention to the problem, we aim to address, we here briefly highlight the main equations presented in the paper (de Felice and Sorge, 2003), till we reach the problem. As a starting point, let us choose the Hamilton-Jacobi equation for the uncharged and spinless particle, but still possessing the magnetic dipole moment in the following form (de Felice and Sorge, 2003):

$$g^{\mu\nu} p_\mu p_\nu - m D^{\mu\nu} F_{\mu\nu} + m^2 = 0 \,, \tag{1}$$

where p_μ is the four-momentum and m is mass of the test particle. The explicit form of the product of polarization tensor $D_{\mu\nu}$ and the electromagnetic field tensor $F_{\mu\nu}$ has shown as

$$D^{\mu\nu} F_{\mu\nu} = 2\mu_\alpha B^\alpha = 2\mu_{\hat{\alpha}} B^{\hat{\alpha}} = -\mathcal{U} \ll m \,. \tag{2}$$

After introducing the new definitions for the radial coordinate r, energy E, angular momentum L, and magnetic parameter:

$$\rho = \frac{r}{2M}\ , \quad \lambda = \frac{L}{2mM}\ , \quad \gamma = \frac{E}{m}\ , \quad \beta = \frac{2\mu B_0}{m}\ , \tag{3}$$

the authors derived the equation for the radial motion in the form (de Felice and Sorge, 2003)

$$4M^2\left(\frac{\mathrm{d}\rho}{\mathrm{d}\tau}\right)^2 = \gamma^2 - V(\rho;\lambda,\gamma,\beta)\ , \tag{4}$$

where τ is the proper time along particle trajectory and the effective potential $V(\rho;\lambda,\gamma,\beta)$ is defined as (de Felice and Sorge, 2003)

$$V(\rho;\lambda,\gamma,\beta) = \left(1-\frac{1}{\rho}\right)\left[1+\frac{\lambda^2}{\rho^2} - \frac{\beta\left(1-\frac{1}{\rho}\right)}{\sqrt{1-\frac{1}{\rho}-4M^2\Omega^2\rho^2}}\right]. \tag{5}$$

where Ω is the angular velocity, measured by a distant observer often called Keplerian frequency, defined as

$$\Omega = \frac{d\phi}{dt} = \frac{u^\phi}{u^t} = \frac{\lambda}{2M\gamma}\frac{\rho-1}{\rho^3}\ . \tag{6}$$

Finally, we arrived at the point where we think the problem is. Indeed, eq. (5) reproduces the effective potential for the neutral particle in the field of the Schwarzschild black hole, if the magnetic coupling is neglected ($\beta = 0$). In the Ref. (de Felice and Sorge, 2003), the authors aimed to discover the effect of the magnetic interaction parameter, β, on the circular motion of the particle. To do so, finding the correct form of the effective potential is crucial, as it is a problem of motion in the central field. As a rule, the effective potential in equation (4) should not depend on the specific energy of the particle. However, if one applies expression of the angular velocity of the particle (6) into the effective potential (5), the effective potential will depend on the specific energy of the particle, as in the case of the paper (de Felice and Sorge, 2003), that contradicts the rule. As proof of that, analogously it is enough to show that the Keplerian frequency (6) is independent of the specific energy (actually, in general, it is independent of both the specific energy and specific angular momentum of the test particle). Therefore, in order to escape from this contradiction, we propose the following procedure:

- In the circular orbit, the particle's four-velocity is given as $u^\mu = (u^t, 0, 0, u^\phi)$ and from the normalization condition of the four-velocity $u_\mu u^\mu = -1$ we obtain

$$u^t = \frac{1}{\sqrt{-g_{tt}-\Omega^2 g_{\phi\phi}}} = \frac{1}{\sqrt{1-\frac{1}{\rho}-4M^2\Omega^2\rho^2}}\ . \tag{7}$$

- On the other hand, since motion of the particle with magnetic momentum in the external magnetic field does not follow the geodesics, by using the (radial) non-geodesic equation for magnetized particle proposed in (Preti, 2004)

$$\frac{D}{D\tau}\left[(m+\mathcal{U})u^{\alpha}\right]=-\mathcal{U}^{,\alpha}\ , \tag{8}$$

we find the following relation:

$$\frac{g_{tt,r}+\Omega^2 g_{\phi\phi,r}}{g_{tt}+\Omega^2 g_{\phi\phi}}=-\frac{2\mathcal{U}_{,r}}{m+\mathcal{U}}\ , \tag{9}$$

where sub-index $_{,\alpha}$ indicates the derivative with respect to coordinate x^{α}. Note that in the derivation of equation (9), u^t is eliminated by using equation (7). Hereafter, applying simple algebraic operations, one can obtain the explicit expression of the angular velocity of the magnetized particle in the circular orbit in the following form:

$$\Omega=\sqrt{-\frac{g_{tt,r}+2g_{tt}\ln(m+\mathcal{U})_{,r}}{g_{\phi\phi,r}+2g_{\phi\phi}\ln(m+\mathcal{U})_{,r}}}\ . \tag{10}$$

One can see from expression (10) that the Keplerian frequency depends only on the spacetime metric and interaction potential, $\mathcal{U}$. In the absence of the interaction term, $\mathcal{U}=0$ or $\mu=0$, the angular velocity of the magnetized particle in the circular orbit, (10), reduces to the well-known Keplerian frequency of the neutral test particle in the Schwarzschild spacetime, $\Omega_0=\sqrt{-g_{tt,r}/g_{\phi\phi,r}}=\sqrt{M/r^3}$.

On the other hand, one has to keep in mind that the interaction potential $\mathcal{U}$ is a function of Ω (See, for example: (de Felice and Sorge, 2003)):

$$\mathcal{U}=\beta\frac{m g_{tt}}{\sqrt{-g_{tt}-\Omega^2 g_{\phi\phi}}}=\beta U(\rho,\beta)\ , \tag{11}$$

and Ω is itself function of magnetic parameter from equation (10).

Before go on further let us introduce the normalized interaction potential $\mathcal{U}\to\mathcal{U}/m$. Now we first find Ω from equation (11) then substitute it into equation (9), and taking into account $\mathcal{U}\ll m$, one can have the following differential equation:

$$2\mathcal{U}_{,r}=-\frac{g_{\phi\phi,r}}{g_{\phi\phi}}-\frac{\mathcal{U}^2}{\beta^2 g_{tt}}\left(\frac{g_{\phi\phi,r}}{g_{\phi\phi}}-\frac{g_{tt,r}}{g_{tt}}\right)\ , \tag{12}$$

or using equations (11) and (3), one can have

$$\beta U_{,\rho}=-\frac{1}{\rho}+\frac{(2\rho-3)}{2(\rho-1)^2}U^2\ . \tag{13}$$

Unfortunately, it is difficult to get an analytical solution for equation (13), however, one can use perturbation in order to obtain a semi-analytical solution for U at least in linear order approximation. Then new interaction potential U can be expanded in the power of β parameter as

$$U(\rho,\beta) = U_0(\rho) + \beta U_1(\rho) + \ldots \tag{14}$$

Substituting it into equation (13), hereafter performing simple algebra one can obtain $U_i(\rho)$ in the form:

$$U_0(\rho) = \left(1 - \frac{1}{\rho}\right)\left(1 - \frac{3}{2\rho}\right)^{-1/2} , \tag{15}$$

$$U_1(\rho) = \frac{(\rho-1)^2}{(2\rho-3)} \frac{U_0'(\rho)}{U_0(\rho)} = \frac{(\rho-3)(\rho-1)}{2(3-2\rho)^2\rho} . \tag{16}$$

Finally, in linear approximation of β parameter, the effective potential for magnetized particle can be written as

$$V(\rho) = \left(1 - \frac{1}{\rho}\right)\left[1 + \frac{\lambda^2}{\rho^2} + \beta\left(1 - \frac{1}{\rho}\right)\left(1 - \frac{3}{2\rho}\right)^{-1/2}\right] . \tag{17}$$

As it is seen, the effective potential for the magnetized particle moving in the field of the Schwarzschild black hole immersed in the uniform magnetic field significantly different from the one presented in (de Felice and Sorge, 2003). Even from this point, one can say that the characteristic circular orbits, such as marginally (i.e., innermost or outermost) stable circular orbits evaluated from the effective potential (17) are different from the ones presented in (de Felice and Sorge, 2003). To estimate these differences quantitatively, let us study one of the most important characteristic circular orbits, such as innermost stable circular orbits. The stability of the circular orbits is guaranteed by the non-negativity of the second derivative of the effective potential with respect to the radial coordinate. If the equality holds then, the solutions of this equation give the innermost stable circular orbits. As this equation has cumbersome form in our case and the solutions cannot be written analytically, we decided to present it in the following form:

$$\beta = -\frac{4\sqrt{2}(\rho-3)\sqrt{\rho}(2\rho-3)^{3/2}}{\rho[5\rho(4\rho-21)+174]-81} , \tag{18}$$

One can easily notice from eq. (18) that in the absence of the magnetic coupling of parameter ($\beta = 0$), one recovers the radius of ISCO of the neutral particle around Schwarzschild black hole ($\rho = 3$). Moreover, in Fig. 1 we demonstrate the dependence of the ISCO radius from the magnetic coupling parameter. One can see from Fig. 1 that for negative values of the magnetic coupling parameter, the stable circular orbits of the magnetized particle are located very far from the black hole, i.e., negative values of the magnetic coupling parameter increases the radius of ISCO. As figure shows, the radius of the ISCO diverges for β tends to -0.8 from right side, $\beta \to -0.8_+$. Thus, from this property we determine that for the existence of the ISCO, the minimum value of the magnetic coupling parameter

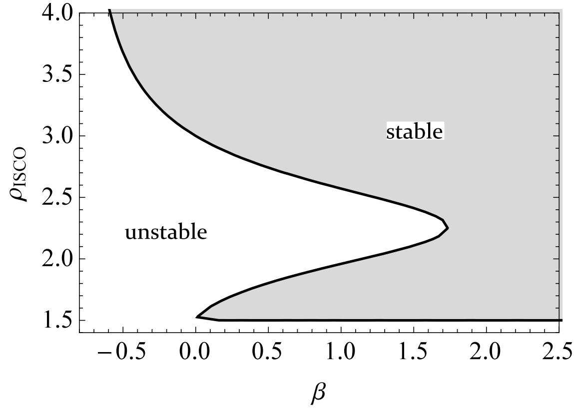

Figure 1. Dependence of the ISCO radius from the magnetic coupling parameter. Where shaded and white regions represent the ones that the stable and unstable circular orbits occupy, respectively.

is $\beta_{min} = -0.8$. On the other hand, for the existence of the ISCO, positive values of β is also restricted. This maximum value is $\beta_{max} = 1.7329$ at which the ISCO is located at $\rho_{\rm ISCO} = 2.2587$. If the value $\beta > \beta_{max}$, the stable circular orbits can exist anywhere of the spacetime outside photonspehere ($\rho > 1.5$).

In this paper, we have presented guidelines on how to derive the effective potential for the magnetized particle orbiting around the Schwarzschild black hole in the presence of the external uniform magnetic field. One has to emphasize that in Ref. (de Felice et al., 2004) the same problem but in the Kerr, spacetime was solved by using the same approach. Later, in Refs. (Preti and de Felice, 2005, 2006) the same approach was applied for the magnetized particle motion around the Schwarzschild and Kerr black holes in the presence of the dipole magnetic field. Now we think that it makes sense if scenarios considered in the papers (de Felice et al., 2004; Preti and de Felice, 2005, 2006) can be recalculated by the method we presented and make comparison with the results shown in them. We keep that calculations for our near future projects.

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